Dynamic Geometry Environment And Its Relation To Thai Students' Higher-Order Thinking
Reasoning In Euclidean Geometry

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DYNAMIC GEOMETRY ENVIRONMENT AND ITS RELATION TO THAI STUDENTS' HIGHER-ORDER THINKING: REASONING IN EUCLIDEAN GEOMETRY

BY

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ABSTRACT

Since its introduction in the late 1980s, Dynamic Geometry Software (DGS) has become one of the most innovative tools in mathematics education. It is defined as graphical software, where geometric figures can be constructed with pre-defined relationships, which will retain when the figures are dynamically manipulated. This digital tool provides a new geometry learning environment inherently different from the traditional paper-and-pencil mode. This research investigates the situation where learners interact directly with this dynamic geometry environment. It examines how learners interpret DGS key features; such as drag-mode and parent-and-child relationship, and how such interpretations relate to their higher-order thinking of reasoning in geometric tasks. Three types of reasoning strategies are pursued in this research. These are: inductive reasoning, deductive reasoning and abductive reasoning. How and to what extent the DGS environment plays a role in the learner's reasoning strategies and arguments is the question at the focus of this research.

Vygotsky's model of tool used as mediated activity and Verillon & Rabardel's Instrumented Activity Situation (IAS) model are used as a framework for this research. These models help to distinguish the independent roles of the learner, the DGS tool, the designed tasks and Euclidean geometry in the overall setting. They also help to clarify the influences that each of these entities may have on each other. The research is conducted in Thailand with a Thai version of The Geometer's Sketchpad to a group of 14-15 year-old lower secondary students. The research method used is a task-based interview, where pairs of students perform geometric construction and exploration tasks with the Geometer's Sketchpad while the researcher challenges their reasoning.

This research finds the tension between the deductive reasoning nature in Euclidean geometry, the inductive nature of visual presentation in the dynamic geometry environment, and the influence of students' experiences in the paper-and-pencil environment on their interpretation of dynamic geometry. Abductive reasoning is found to be the students' main reasoning strategy, with a combination of inductive and deductive reasoning to support their verification of the hypotheses.

Keywords: *Dynamic Geometry, Educational Software, Euclidean Geometry, Reasoning, Technology in Education, Instrumented Activity Situation model*

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1 INTRODUCTION

This introductory chapter provides the background concept, selected research setting, research aim as well as the outline of the thesis structure. It gives an overview of the foundational idea and the justification of Thailand as the location setting of the research.

1.1 RESEARCH BACKGROUND

One of the most significant recent innovations in school education is the incorporation of digital technology into teaching and learning activities. During the past decades, various forms of digital technology have been integrated into different areas of school education. They include programming environments (LOGO, Visual Basic), software applications (word processor, spreadsheet, PowerPoint), communication tools (e-mail, instant messenger, web-board forum, social networking), information resources (internet, e-learning), content-free environments (CAD/CAM, CAS, dynamic geometry) and tutorial packages.

In the area of mathematics education, effective integration of digital technology has been a topic of interest among researchers, scholars and educators worldwide and was presented as a main theme for the 17th International Commission on Mathematical Instruction (ICME) study in 2006. In a recently published report of the study, Hoyles and Lagrange (2010) provide an overview of digital technology development and what had been studied on the impact of technology on teaching and learning mathematics since the very first ICME study in 1986. The summary of the study exhibits a broad range of areas in which technology can play a role in mathematics education. They include the design process of the learning environment, learning trajectories and automatic assessment, means of communication and collaboration, teacher's pedagogic strategies as well as issues on curriculum revisions. Despite such wide possible impacts digital technology may have on mathematics education, its common benefits addressed by a number of participants of this study are flexibility and efficiency.

Sacristan et al. (2010) report that digital technologies have opened up diverse trajectories for learners to construct and comprehend mathematical knowledge, which may not necessarily correspond to the teacher's hypothetical learning trajectories. Technology, therefore, broadens pedagogical flexibility, which may better match the learner's diverse learning styles. While, Olive et al. (2010) identify key benefits of digital technology in mathematics learning as a facilitator to deal with complex and massive mathematical operations, a motivator to engage learners with active activities and a provider of automatic feedback, giving an immediate reflection on the performance. Nevertheless, all these impacts of technology remain at a level of enhancement tool. Most of its benefits are not fundamentally distinguishable from non-technological settings. The authentic roles of technological environments on the learning process are scantily touched upon, and as suggested by the report's contributors, leaving room for further research (Drijvers, Mariotti, Olive, & Sacristán, 2010; Olive, et al., 2010; Sacristán, et al., 2010). This research, therefore, targets to investigate this gap by examining the distinct role of technological environment on learners' thinking and learning processes. The dynamic geometry software which provides the Euclidean geometry environment in a novel way is selected as a technological environment with learners' geometric reasoning skill set as a focus in this research.

1.2 THAILAND AS THE RESEARCH SETTING

Though research about learners' reasoning processes in the dynamic geometry environment is universal research that can be conducted anywhere, this research is conducted in Thailand for the following reasons:

This research is supported and sponsored by The Institute for the Promotion of Teaching Science and Technology (IPST)-Thailand, so setting this research in Thailand benefits the mission of IPST better than anywhere else. Moreover, as the researcher, I am also a Thai citizen with a native background in that country and understand the cultural context of this research fully, thereby reducing the possibility of a gap in the research which may occur if it is conducted in a foreign culture. Another reason stems from the fact that language is an important mediating tool for the process of reasoning. The researcher will understand the learner's mental process of reasoning best if he/she shares the same language. Conducting this research in Thailand where the Thai language as the native language of both the researcher and the subject learners is used, already facilitates the empirical and analysis process of the research.

Moreover, Thailand is a country that actively promotes the use of digital technology in school mathematics education. The B.E. 2544 Basic Education Mathematics Curriculum (IPST, 2001) clearly states that technology should be used to develop the learner's mathematical skill, especially in problem solving, though there is no clear indication of what and how technology should be used. Apart from the mathematics content strands, this curriculum also highlights a dedicated strand for mathematical skills and processes which involves the cultivation of higher-order thinking. This curriculum identifies 'higher-order thinking' as problem-solving, reasoning and creativity. Despite a lack of clear connection of how technology should be used to develop such higher-order thinking, it is still obvious that Thailand's mathematics curriculum gives importance to both issues in mathematics education.

1.3 THE GEOMETER'S SKETCHPAD AS THE DYNAMIC GEOMETRY ENVIRONMENT

Thailand recently made an effort to integrate technology into mathematics education by an acquisition of a national right of a dynamic geometry system called 'The Geometer's Sketchpad' in 2004, followed by a project to localise the software into Thai language for school use. Dynamic Geometry System (DGS) is graphical software where geometric figures can be constructed and dynamically manipulated. These figures are constructed from geometric relationships and their properties will retain when using a mouse to drag or move any part of the figures (Bloomfield, 1992; Olive, 2000; Sträßer, 2002). Nevertheless, the true purpose of DGS is not simply a geometric construction tool. Its key feature, where the user can use a mouse to modify the figure while retaining all its pre-defined geometric properties called 'drag-mode', allows learners to experiment and observe the variant and invariant properties of the geometric construction. This feature encourages learners to pose and try to validate geometrical hypotheses. Many scholars, therefore, considered DGS as a computerised Euclidean geometry environment where learners can interact in dynamic mode (Bloomfield, 1992; Goldenberg & Cuoco, 1998; Green, 1992; Weaver & Quinn, 1999). Since the only officially approved Thai version of dynamic geometry software available in Thailand is The Geometer's Sketchpad (GSP) Thai Version (Jackiw, 2005). This research, therefore, needs to use this particular software as the environment.

Localising educational software such as The Geometer's Sketchpad from the English version to the Thai is not a straightforward process. Though the software's strong reference to the universal mathematical language and traditions of Euclidean geometry, coordinate geometry, transformational geometry and algebra may lessen the cultural issues commonly found in many localisation projects, the fundamental difference between English and Thai sentence structures still demands a significant reorganisation of software strings in order to make the translations sensible to Thai users. As a member of the localisation team of the Thai Geometer's Sketchpad, I have been thoroughly involved in this transformation process and witnessed the endeavours of all concerned to overcome such a challenge. The most common problem in the translation process is the absence of the plural structure in the Thai language. All the English strings that refer to plural objects such as '6 points are selected' need to be restructured so that 'ได้เลือกจุด 6 จุด' can be displayed in the Thai version. Nevertheless, the Thai localisation team worked closely with the software developer in order to ensure that the Thai version displays all the string variables correctly. Apart from this language issue there is no other major problem in the localisation process. The Geometer's Sketchpad Thai version, therefore, presents a thoroughly localised English version without the need to omit or modify any part or feature.

1.4 HIGHER-ORDER THINKING AND REASONING IN THE THAI CULTURAL CONTEXT

With higher-order thinking and reasoning as central concepts of this research, it is appropriate to discuss how Thai society in general perceives these concepts before certain aspects are identified as foci of this study.

Thailand considers itself to be a deep-rooted Buddhist country. Most of its cultural formation is therefore shaped by the Buddhist philosophy. For most Thai people, higher-order thinking is considered a requisite transcendental skill and a practice one needs to learn in order to abolish all life's sufferings, leading to the superior goal of all Buddhists, i.e. achieving 'nirvana'. To learn how to extinguish passion, the main source of suffering, a Buddhist needs to reason for him/herself. The approach of reasoning in Buddhist teaching emphasises individual experience and wisdom with the ultimate aim of analysing the source of human yearning and passion. Buddhist reasoning is therefore a primary tool for rationalising human feelings. Individual reasoning is reinforced through the famous Buddhist teaching of reasoning prejudices that all Buddhists should avoid called 'Kalama Sutta'. This teaching lists 10 common prejudices that usually lure us to an unintelligible conclusion. It warns us not to believe in certain knowledge simply because:

a) it is being repeated by many

b) it is an ancient tradition

c) it is widespread news

d) it is in a scripture

e) it is surmised

f) it is an axiom

g) it is implied by outer appearance

h) it conforms to our presupposition

i) it is said by the respectable

j) it is said by our teacher

(Punyanupap, 2003)

Instead, one should believe in something after one contemplates and personally realises on his/her own the cause and consequence of such subject. (It is interesting to note that the word 'reason' in Thai is literally translated into English as 'cause-consequence'.) Buddhist reasoning, therefore, advocates personal experience of the subject. It values individual justification based on inner reactions to the subject of interest.

Despite such a strong religious foundation in the Thai culture, Thai society also realises the need to modernise itself by adopting western traditions and cultures to foster the nation's economic development in this era of globalisation. Mathematical, scientific and technological education is a good example of how Thai society imports western civilisation into its context. With very little knowledge of geometric shapes and measurements, the majority of mathematical education content in Thailand is imported and the students realise this very easily by the foreign terms they persistently encounter in mathematical lessons. The term 'higher-order thinking' used in the B.E. 2544 Basic Education Mathematics Curriculum (IPST, 2001) is actually borrowed from the common core standard of the US National Council of Teachers of Mathematics. The processes of mathematical reasoning, i.e. inductive and deductive reasoning, are also foreign to Thai students, though Thai mathematical educators have managed to invent specific Thai terms for these types of reasoning. The concept of 'higher-order thinking' and traditions of mathematical reasoning is, therefore, clearly distinguishable from the Buddhist concept of 'higher-order thinking' and reasoning. The possibility for confusion is virtually non-existent.

1.5 THAI LANGUAGE AND REASONING

With Thailand as the cultural context and Thai as the communication language for this research, issues about the relationship between the Thai language and western reasoning may arise, in particularly to what extent the Thai language can accommodate the process of western reasoning. The Thai language is sufficiently rich so as to convey the nuance of the process of reasoning, as well as other languages, especially English. The Thai language has specific terms with exactly the same meanings as English words in relation to reasoning, for example 'เหตุผล' for 'reason' / 'เพราะว่า' for 'because' / 'ดังนั้น' for 'so' or 'therefore'/ 'ทำไม?' for 'why?' / 'เพราะอะไร?' for 'for what reason?' / 'ไร้เหตุผล' for 'not making sense' or 'น่าจะ' for 'possibly' or 'perhaps'. Thai language also has its own technical terms invented by Thai mathematicians for particular types of reasoning: 'inductive reasoning' – 'การให้เหตุผลแบบอุปนัย', 'deductive reasoning' – 'การให้เหตุผลแบบนิรนัย' and 'abductive reasoning' – 'การให้เหตุผลแบบจารนัย' to be further discussed in the next chapter though these terms are not commonly used in everyday life. In terms of syntax and grammar, the Thai language uses a clear and straightforward structure in a reasoning sentence and it is rather convenient to distinguish illogical reasoning from misuse of language. The syntax of the

Thai language is sufficiently rudimentary so as to separate the anomaly in the content and use of the language. With such subtleties in the Thai language, it can convey a wide range of reasoning processes as effectively as other languages, such as English.

1.6 RESEARCH AIM

The general aim of this proposed research focuses on the further examination of how the use of digital technology in the learning environment relates to the learner's cultivation of higher order thinking especially in reasoning. The choice of digital technology studied, is the Geometer's Sketchpad system, which is currently used in many schools in Thailand. The mathematical topic involved is Euclidean Geometry, which is included as one of the main content strands of Thailand's mathematics curriculum. The outcome of this proposed research is expected to outline the possible relationships between the dynamic geometry environment and students' higher-order thinking especially in reasoning.

1.7 THESIS OUTLINE

This thesis comprises 11 chapters; this INTRODUCTION chapter is followed by CHAPTER 2: LITERATURE REVIEW where relevant issues and research studies will be discussed. The concept of Instrumented Activity Situation model and its influencing ideas which will be used as a framework for this research is discussed in CHAPTER 3 entitled THEORETICAL FRAMEWORK. CHAPTER 4 is identification and elaboration of the RESEARCH QUESTION as a central query of this study. The organisation of the research will be outlined in CHAPTER 5: RESEARCH DESIGN. CHAPTER 6: DEVELOPING ANALYSIS MODEL refines the model of study used in the data analysis process. The data analysis section comprises four dedicated chapters, i.e. CHAPTERS 7-10. Each will discuss the data collected from different perspectives. The summary of findings will then be presented in the final CHAPTER 11: CONCLUSION where the implications and direction of future research will also be given.

2 LITERATURE REVIEW

This chapter is divided into nine sections. It begins with the discussion of three key concepts relevant to this research, i.e. 'higher-order thinking', 'reasoning' and 'argumentation'. Subsequently, the different roles digital technology may have in education are explored. The examination of the Dynamic Geometry System and its key features will follow, and the chapter concludes with a survey of research studies regarding the role of DGS on the learner's geometrical reasoning process.

2.1 HIGHER-ORDER THINKING

The term higher-order thinking clearly indicates the hierarchy and variety of the thought process. Though there is no generally accepted definition of what 'higher-order thinking' is, many scholars have discussed this concept through the ideas of human intellectual development. This section surveys different notions of 'higher-order thinking' through the works by Plato, Piaget, Vygotsky and Bloom. The common characteristics of 'higher-order thinking' are then identified.

2.1.1 Plato's Levels of Thinking

The idea of dividing the human mind into lower- and higher- order stages can be traced back to Greek times in the dialogues between Plato and Socrates found in *The Republic* (Lee, 1974). Plato separated the human mental state into two main levels namely, 'opinion' and ‘knowledge’. The lower ‘opinion’ level was further divided into ‘illusion’ and ‘belief’. ‘Illusion’ includes a variety of second-hand impressions without direct connection to the real object. While ‘belief’ refers to common-sense beliefs on physical and moral matters, generated from direct interaction with a real object. The higher ‘knowledge’ level is also further divided into two sub-levels; ‘reasoning’ and ‘intelligence’. ‘Reasoning’ covers the procedure of mathematical deductive reasoning, while ‘intelligence’ accounts for full understanding culminating in the vision of the ultimate truth. The diagram of Plato’s mental state classification is depicted in Figure 2.1.

KNOWLEDGE	INTELLIGENCE: full understanding of the ultimate truth	Higher
	REASONING: mathematical deductive reasoning	
OPINION	BELIEF: common-sense beliefs, and direct contact with objects	
	ILLUSION: second-hand impression, no direct contact with objects	Lower

Figure 2.1 Plato's levels of the human mental state

From the descriptions of these four levels of mental state, the criterion used to separate ‘opinion’ from ‘knowledge’ is the extent to which such mental state can claim the truth. ‘Illusion’ and ‘belief’ are based on guessing without experience and common-sense respectively. Hence, they give weak credibility to the claim, while ‘reasoning’ relies on the more logical process of verification by deduction, which eventually leads to ‘intelligence’ or the full understanding of the ultimate truth.

The clear division of Plato's higher and lower order stages of thinking, therefore, is the threshold between 'belief' and 'reasoning' levels. The distinguishing feature of these two levels of thinking is the move from concrete thinking as in the 'belief' stage to the abstract thinking of 'reasoning'. Thus, for Plato, deductive reasoning can be used as evidence of a person's transition from lower to higher order thinking stages.

2.1.2 Piaget and Higher-Order Thinking

Though the term 'higher-order thinking' was not directly used by Piaget himself, the concept can still be inferred from his Theory of Cognitive Development. Piaget's Theory of Cognitive Development divides human intelligence growth into four distinct stages, based on the common intellectual capabilities appearing at different broad periods of development (Piaget, 1972). These stages are:

1) Sensori-motor stage (age 0-2) where experience is gained through basic physical senses.
2) Pre-operational stage (age 2-7) where children begin to use symbol, language and mental imagery.
3) Concrete-operational stage (age 7-11) where concrete logic starts to develop.
4) Formal-operational stage (age 11 onwards) where children can think and reason in abstract.

From this theory, the first two stages, i.e. Sensori-motor stage and Pre-operational stage may be considered to be lower-order thinking while the last two stages, i.e. Concrete-operational and Formal-operational stages may be considered to be higher-order thinking.

Piaget based his Theory of Cognitive Development from the close observation of the real mental growth of children from birth to early adulthood. The age range specified for each stage is provided as a broad period for reference only. Piaget made the point himself that different children may reach each stage at different ages, possibly beyond the given range, depending on maturity of the nervous system and experience in the social environment. However, the order remains constant (Piaget, 1971).

In the Concrete-operational stage, Piaget states that the child can think in a logically coherent manner about objects that do exist and have real properties about the actions possible (Sinclair (1971) as cited in Gallagher (1981)). One of the key characteristics of this stage of thinking is an understanding of causality, where the former influences the latter with respect to the attribution of form to content, and vice versa (Piaget, 1972). While the key characteristic of the Formal-operational stage of mental ability is the capacity to deal with hypothesis instead of simply with objects (ibid.). This capacity is determined by strategies adopted by children to attack the problem, which Piaget defined as deductive reasoning methods (Piaget, 1977). Children at the Formal-operational stage can use hypothetical-deductive reasoning, i.e. they can deduce systematically from available data to draw a conclusion. They are also capable of devising a plan logically to achieve the goal, while the children at the Concrete-operational stage will rely more on the empirical feature of the task and lack the skill to plan a solution to the problem (Piaget, 1972). The presence of hypothetical strategy in a problem-solving activity can, therefore, be used as an indication of children's maturation to the Formal-operational stage.

The distinction between the Concrete-operational stage and the Formal-operational stage in Piaget's Theory of Cognitive Development also reflects the separation between 'belief' and 'reasoning' in Plato's model of mental stages. They both consider that the ability to think in abstract is a step higher than concrete-thinking. However, Piaget's reference to 'logic' in the Concrete-operational stage indicates a different kind of reasoning, based on concrete evidence which generally conforms to the 'belief' stage in Plato's model. Piaget's Theory of Cognitive Development, therefore, highlights two different forms of 'reasoning', separated by their concrete and abstract natures.

One curious difference between Plato's and Piaget's ideas about the development of human thinking is the age when the ability to think in abstract is acquired. While Piaget's Theory of Cognitive Development indicates that abstract thinking will emerge at the age of early puberty, Plato suggests that one has to wait until the age of 30 years old before abstract thinking becomes mature (Lee, 1974). In Plato and Socrates' dialogues on a curriculum to train the philosopher rulers, they divided the stages of education into four phases. The first phase involves reading and writing lasting until a person is 18 years old. Following that, two more years are dedicated to physical training before a series of mathematical discipline training during the 20s. Once a person is 30 years old, they will then be ready for education on dialectics which is the exercise of pure thought requiring the highest level of intelligence. Moreover, not every person will be able to go through these steps since each stage undergoes a process of selection for the more able. These descriptions of human mental development reflect the different general capacity of people in ancient Greece, which may no longer be applicable to the contemporary practice. Nevertheless, the strict order of the training subjects outlined in the *Republic (*ibid.*)* still implies the hierarchical development of human thinking where concrete education in mathematics is considered a necessary foundation for the abstraction of dialectics.

Despite a strong influence in the field of cognitive psychology, Piaget's Theory of Cognitive Development has also become a subject of criticism among many scholars. One of the major arguments is that Piaget's theory neglects the role of knowledge domain, where some research studies show that children do not reach a certain stage of thinking at a certain time but will achieve a certain stage separately across the knowledge domains (Sutherland, 1992). Hence, the model of mental progress is domain-specific, and one cannot simply conclude that a child is currently at this particular stage at a certain point of time. Bryant and Trabasso (1971) also reveal that Piaget's use of language to communicate with children in his studies can play a significant role in his interpretation of children's responses. They demonstrate that, with a careful use of language, some young children may achieve a stage of mental development higher than as purported by Piaget. This fact reduces the trustworthiness of Piaget's experiments and his theory.

Another major concern is Piaget obviously ignores the issues of social class, individual differences and personality differences which may affect their mental capacity, while many studies, especially about an IQ test, illustrate the divergence of results even among children of the same age group (J. Cohen, 1983). Moreover, Piaget's theory merely outlines the steps of intellectual progress without a deeper explanation of what is actually going on during the transition to a higher stage (Sutherland, 1992). This theory, therefore, does not clarify much about these mental phenomena. Nevertheless, the hierarchical order of stages as presented by Piaget has been virtually unchallenged. Piaget's identification of developmental stages is, therefore, generally accepted, though the details of age ranges and the generalisability to a wider population may remain arguable.

2.1.3 Vygotsky and Higher-Order Thinking

Vygotsky gives different views on children's higher-order thinking through his identification of human 'Higher Mental Function'. Unlike Piaget, Vygotsky places greater emphasis on language as an essential factor in a child's development of higher-order mental capability. Referring to behavioural studies by Bühler (1930) and Köhler (1973) as cited in Vygotsky (1986), Vygotsky considers the mental development of new-born children during their first year as indifferent from higher animals such as chimpanzees. Though there is evidence that children at this early age naturally adopt the action of physical tool use, i.e. an application of a tangible object as a device to achieve a desired goal, their use of tool shares the same characteristic with those appearing in higher apes (Vygotsky, 1986). It is the stage where children begin learning to use language in order to communicate with each other that Vygotsky deems a departure from animal behaviour to human behaviour. He identifies communicative language as a 'psychological tool' or an artificial formation directed toward the mental process. He also claims that while the use of a physical tool is genetically inherited in an organism's innate natural mental ability, the psychological tool is a product of social interactions and, therefore, an exclusive human activity (ibid.).

Language is the typical example of a psychological tool as defined by Vygotsky. Besides being a communication tool, language is also an indispensible mediating tool for the human advance thinking process (Vygotsky, 1986). With language as a key factor, Vygotsky separates human mental activities into 'Higher Mental Function' and 'Lower Mental Function'. 'Higher Mental Function', therefore, refers to intellectual activities which are mediated by language, while 'Lower Mental Function' includes the pre-language application of a physical tool, as well as the use of language merely for communication (Vygotsky, 1978). Vygotsky mentions four major criteria to distinguish between these two levels of mental functions. These differ by 1) origins, 2) structure, 3) the way of functioning and 4) the relation to other mental functions (Subbotsky, 2010).

As for origins, Vygotsky considers 'Lower Mental Function' to be genetically inherited, burgeoning from instinct, while 'Higher Mental Function' is a product of social interaction which will not take place without interaction with others. In terms of structure, 'Lower Mental Function' is deemed to be a direct process between subject and object. While the connection between subject and object mediated by a psychological tool such as language is deemed as 'Higher Mental Function'. The next criterion of the function concerns the intention of the subject. The 'Lower Mental Function' is involuntary, while the 'Higher Mental Function' is always voluntary. The final distinction between 'Lower Mental Function' and 'Higher Mental Function' is their relativity to other mental functions. While 'Lower Mental Function' operates in isolation, 'Higher Mental Function' has a close link to the broad system of other mental functions (ibid.). To highlight his socio-cultural perspective of intellectual development, Vygotsky explicitly claims "*the very mechanism underlying higher mental functions is a copy from social interaction. All higher mental functions are internalised social relationships*" (Wertsch, 1985). From this elaboration of these two levels of mental functions, the key factor that distinguishes 'Higher Mental Function' from 'Lower Mental Function' is the internalised mediation of language as a psychological tool to achieve certain goals voluntarily, and dependent on the overall system of mental function.

Despite the intricate description of these levels of mental functions, Vygotsky has never made clear what type of skills should belong to 'Lower Mental Function' and to 'Higher Mental Function'. He briefly defines 'Higher Mental Function' as functions of voluntary attention, logical memory, reasoning, and rational, volitional, goal-directed thought (Minick, 1996; Ravenscroft, 2001), stating the skills of remembering, comparing, reporting, and choosing them as examples of 'Lower Mental Function' (Vygotsky, 1978). It can only be inferred that skills which need language to execute should belong to the 'Higher Mental Function'. The prime purpose of Vygotsky's distinction between 'Lower Mental Function' and 'Higher Mental Function' is, therefore, to identify human higher intellectual activities from the lower mental activity of animals.

Vygotsky's proposed notion about Higher Mental Function has been criticised for his lack of consideration of human elementary mental processes, as well as the negligence of the way the subject grasps the syntax of language, which can play an essential role, especially when it is used as a mediating tool (Johnson-Laird, 1986). Confrey (1995) also comments that Vygotsky's strong emphasis on the role of language in the process of knowledge acquisition may devalue the importance of concrete activity, which can also play an important part in the mental development process. Moreover, Vygotsky's identification of language as an indispensible tool for 'Higher Mental Function' also rules out other possible non-verbal means such as visualisation, spatial sense or aesthetic judgement in art and music from higher level intellectual activities. However, these criticisms mostly concern pertinent issues, which should be considered in order to elaborate Vygotsky's theory. They do not really challenge his central concept of language as a mediating tool for 'Higher Mental Function' or its socio-cultural basis.

2.1.4 Bloom's Taxonomy of Educational Objectives

The origin and purpose of Bloom's Taxonomy of Educational Objectives may be different from those theories by Piaget and Vygotsky. This taxonomy is developed as a frame of reference for teachers, instructors and educators to facilitate communication in their tests and assessment materials. It was designed to be a classification of student behaviour which represents the intended outcomes of the educational process (Bloom, 1956). The taxonomy comprises three different domains:

1) Cognitive Domain.

2) Affective Domain.

3) Psychomotor Domain.

However, the domain that closely relates to higher-order thinking is the cognitive domain since it involves knowledge, comprehension and critical thinking, while the affective domain places its focus on the learner's emotional reactions and the psychomotor domain aims to assess their physical abilities. Bloom classifies objectives in cognitive domains into six classes in the following hierarchical order. They are:

1) Knowledge Class, which simply measures the ability to recall learned facts and knowledge.

2) Comprehension Class, which looks for the apprehension of concept communicated in isolation without reference to other materials.

3) Application Class, which focuses on the ability to use general ideas or rules of procedures in particular, and concrete situations.

4) Analysis Class, or the breaking down of a communication into its constituent elements, in order to make clear the relative hierarchy.

5) Synthesis Class, which involves the ability to put together the elements and parts to form a whole.

6) Evaluation Class, which concerns judgements of values both by internal evidence or external criteria, including the ability to assess the accuracy of facts, statement or proofs and to indicate the logical fallacies in arguments.

These proposed classes of educational objectives are an outcome of a compilation and analysis of actual test materials used by teachers across various fields of studies. The distinctions between each class are intended to reflect the teachers' distinctions of each test's objective. The hierarchy of classes is ordered from the simplest to the most complex objectives, i.e. knowledge is the simplest objective, while evaluation is the most complex objective. This order came from the statistics of students' performance in tests used in the project. The test items which more students can answer correctly are considered simpler, and vice versa. The data also suggests that the objective in the higher level usually requires the objective in the lower level. Hence, though Bloom (1956) did not explicitly assert the rigidity of order, the proposed hierarchy still implies the step of levels. Despite the original purpose being an educational framework for teachers and educators, Bloom's Taxonomy of Educational Objectives, to some extent, also reflects the scheme of the learner's intellectual development.

Bloom's Taxonomy of Educational Objectives has been subjected to much criticism since its introduction in 1956. One of the Taxonomy's pitfalls was highlighted by Bloom himself, where he accepted that the major problem of this taxonomy was that it is necessary in all cases to know the nature of the examinees' prior educational experiences. This is because different examinees may be able to solve the same problem but with different skills, depending on their experience (ibid.). Sockett (1971) and Pring (1971) give a stronger critique on the taxonomy's lack of concern about 'content', making the terms: remember, comprehend or apply, meaningless. They both claim that these behaviours need content to support its meaning. The content-free nature of the taxonomy, therefore, makes it an inadequate framework. Furst (1981) points out that the hierarchy of the taxonomy can also be questionable, as it is not always the case that the skill in the lower level will be simpler than the skill in the higher level, e.g. certain demands for Knowledge are more complex than certain demands for Analysis and Evaluation when content is taken into consideration. He even proposes that the classes should be presented as parallel rather than linear hierarchical. Hirst (1974) also complains about the comprehensiveness of the taxonomy where the progress of rational understanding, the goal of acquisition of knowledge and rational belief are absent from the taxonomy.

Based on this critique of the original taxonomy, Anderson and Krathwohl (2001) revise the taxonomy by changing the terms from noun to verb, and by adjusting some words used, e.g. Synthesis is changed to Create. They also swap the order between Evaluate and Create since Evaluation is often a necessary part of the precursory behaviour before Create. Moreover, the first class was changed from Knowledge to Remember, and the Knowledge was transformed into another dimension of taxonomy by classifying it into Factual, Conceptual, Procedural and Metacognitive knowledge, generating a tabular relationship between knowledge dimension and cognitive process dimension. Nevertheless, the question about the hierarchy still persists as Anderson & Krathwohl continue to emphasise the sequential order of both the Cognitive Process Dimension and Knowledge Dimension by referring to empirical evidence from the original taxonomy.

Bloom's Taxonomy, both in its original and revised forms, though giving a much clearer classification of intellectual skill, is still subjected to a hierarchical problem which makes it difficult to clearly distinguish between higher-order and lower-order thinking skills, especially when its construction was not actually based on any theory of cognitive development. The tradition that divides Bloom's three lower classes of cognitive domain as 'lower-order thinking', and the remaining three higher classes as 'higher-order thinking' (Forehand, 2005; Jolliffe & Ponsford, 1989; Pogrow, 2005), lacks strong theoretical grounds to justify the validity.

2.1.5 Identifying Higher-Order Thinking

As can be seen from these different notions of stages in the development of human thinking, there is no general agreement of what exactly should be classified as higher or lower levels of thinking. Plato's concept of the human mental state identifies 'higher-order thinking' by distinguishing the lower concrete common-sense belief from the higher abstract deductive reasoning. Piaget considers logical thinking, both in concrete and abstract, as higher-order thinking and separates this from the lower level of mere use of language and symbol. Vygotsky defines 'Higher Mental Function' as those human activities which require internalised language mediation. Bloom's taxonomy of educational objects, which suggests the hierarchical order of mental skills, is still disputable.

Despite the varied opinions about the definition of higher-order thinking from these scholars, they all identify 'logical reasoning' as a higher mental skill. Plato and Piaget also distinguish concrete reasoning from abstract reasoning, and claim the hierarchical development from the former to the latter. It can be rationally concluded that logical reasoning is one of the generally accepted mental activities belonging to higher level thinking, and will be selected as the main skill in this research. Apart from reasoning, the relevant issues from the discussion of 'higher-order thinking' includes the classification of reasoning, especially concrete and abstract reasoning, the age ranges in stages of mental development, the importance of language as a mediation tool for reasoning and the content domain of reasoning skill.

2.2 REASONING

Reasoning is defined as a human cognitive that involves deriving inferences from principles or evidence, whereby the individual either infers new conclusions or evaluates proposed conclusions from what is already known (Johnson-Laird & Byrne, 1993). From this definition, the purpose of reasoning is to construct a new rule or conclusion based on known rules or principles and/or evidence.

Reasoning is generally distinguished into three types, namely, deductive reasoning, inductive reasoning and abductive reasoning (Hanson, 1965). Each type of reasoning is described and explained as follows:

2.2.1 Deductive Reasoning

In deductive reasoning, the premises and conclusion of an argument are related in such a way that the truth of the premises guarantees the truth of the conclusion, i.e. if the premises of the argument are all true, then the conclusion must be true (Salmon, 2007). A classic example of deductive reasoning is Aristotle's syllogism: an argument consisting of three parts, a major premise, a minor premise and a conclusion (Russell, 2004).

Major premise:	All men are mortal.
Minor premise:	Socrates is a man.
Conclusion:	Socrates is mortal.

From the example of deductive reasoning given above, it should be noted that the conclusion does not give any new information other than that already contained in the premises. Moreover, it is subjected to the problem of 'posterior analytics', which questions: If the truth of the conclusion always depends on the truth of the premises, how is the truth of the very first premise obtained? (ibid.). 'Posterior analytics' emphasises the fundamental weakness of deductive reasoning, where the very first premise is usually an agreement of a subject's community. This means that an individual may or may not accept such agreement as a truth. All deductive reasoning is, therefore, based on an assumption that the first premise is true or universally accepted. The major premise 'All men are mortal' given in the above example can still be questioned as to its truthfulness since it is theoretically possible that there is an immortal man we are not aware of.

An example of deductive reasoning in geometry is given below.

Premise 1:	The sum of the internal angles of any triangle is always 180 degrees.
Premise 2:	An equilateral triangle has three equal internal angles.
Conclusion:	Each internal angles of an equilateral triangle is 60 degrees.

From this example, the first premise is true only in the case of Euclidean geometry. This first premise and the deduced conclusion would not be true for non-Euclidean geometry, where the sum of the internal angles may be more or less than 180 degrees, and the internal angles of an equilateral triangle may not be 60 degrees. This circumstance shows the sensitivity of deductive reasoning where the validity of the conclusion entirely relies on the acceptance of the premises in different situations.

2.2.2 Inductive Reasoning

Inductive reasoning, by contrast, is a process whereby regularities or order are detected and, inversely, whereby apparent regularities, seeming generalisations, are disproven or falsified. It takes place by detecting commonalities through a process of comparing both similarities and differences (Klauer, 1996). The truth of the premises in inductive reasoning, therefore, does not guarantee the truth of the conclusion as in deductive reasoning (provided the truthfulness of the premise is accepted). Inductive reasoning, therefore, represents that a statement has a certain degree of probability (Munoz-Colberg, 1977; Salmon, 2007). Nevertheless, it gives more information than is already contained in the premises, and hence, better contributes to the generation of new knowledge than deductive reasoning. The common example of inductive reasoning is given below:

Premise: The sun has risen in the east every morning until now.

Conclusion: The sun will always rise in the east.

Since inductive reasoning involves an examination of multiple evidence of the same case, the issue arises regarding how extensive the evidence needs to be before we can reach a conclusion. The more evidence, the stronger the inductive claim, especially a confirmative claim (while single negating evidence is sufficient for falsification). This leaves room for a different type of inference where one reaches a conclusion from a single piece of evidence. For example, when one is instructed that a particular flower is called a rose, he/she may immediately assume that a rose must be red by seeing just one example. The conclusion is, therefore, drawn by a partial property of the object which may or may not be true. This particular type of inference from just a single case should then be distinguished from inductive reasoning and may be called ‘naïve empirical inference’ (see also Balacheff, 1988).

An example of inductive reasoning in geometry is given below:

Premise: The ratio of the diameter and circumference lengths of circles C_1, C_2, C_3, …, C_{100} lies between 3 and 4.

Conclusion: The ratio of the diameter and circumference lengths of any circles should also lie between 3 and 4.

It can be seen that this inductive conclusion is entirely based on possibility from past experience. After calculating the ratios of the diameter and circumference lengths of 100 circles, it is already concluded that the ratio would lie between 3 and 4 for any circle. Though the conclusion is highly likely, there is no logical explanation of why it is so. Inductive reasoning can therefore give us a certain degree of confidence but could not guarantee it to be the whole truth.

2.2.3 Abductive Reasoning

Apart from deductive reasoning and inductive reasoning, Charles Sanders Peirce proposes another mode of inference called 'abductive reasoning', and defines it as *"a process of studying facts and devising a theory to explain them"* (Frankfurt, 1958, p. 593). Peirce calls such devised theory a 'hypothesis' and summarises that *"abduction is the process of forming explanatory hypothesis"* (Frankfurt, 1958, p. 593). He also distinguishes abductive reasoning from deductive reasoning by their different natures of inference. While deductive reasoning is explicative in nature, i.e. the conclusion simply explicates what is stated in the premises, abductive reasoning is ampliative, i.e. the conclusion amplifies the given premises by hypothesising similarly to the case of inductive reasoning (Fann, 1970). He then further distinguishes abductive reasoning from inductive reasoning by pointing out that inductive reasoning is an inference from particulars to a general law, while abductive reasoning is an inference from a body of data to explaining a hypothesis (ibid.). To illustrate abductive reasoning, Peirce presents a logical form of abduction reasoning as follows:

> *"The surprising fact C is observed.*
>
> *But if A were true, C would be a matter of course.*
>
> *Hence, there is a reason to suspect that A is true."*
>
> (D. R. Anderson, 1986, p. 156)

Or to give a real-life example:

> This watermelon is yellow.
>
> All watermelons from Thailand are yellow.
>
> Hence, there is a reason to believe that this watermelon is from Thailand.

In essence, Peirce's abductive reasoning is an inference by means of hypothesis formulation, suggesting that something *may* be true, while deductive reasoning proves that something *must* be true, and inductive reasoning shows that something *actually is* operative (Fann, 1970).

An example of abductive reasoning in geometry is given below:

The two diagonals of any rhombus are always perpendicular to each other.

A rhombus can be viewed as two congruent isosceles with one of the rhombus diagonals as their bases.

Hence, there is reason to believe that the property of the perpendicular line with the base through the top vertex halves the base involves the rhombus' perpendicular diagonals property.

From this example, a surprising fact that two diagonals of any rhombus are always perpendicular to each other is observed, and there is a need to verify the reason for this phenomenon. With the knowledge about the properties of a rhombus and an isosceles, one may realise that the fact a rhombus can be viewed as two congruent isosceles sharing a base, and the property of the perpendicular line from an isosceles' base passing the vertex may be used to deduce such phenomena. Though this identification of known properties might not be logically sufficient to explain the phenomenon, such hypothesising can be a useful starting point for the process of further verification.

Peirce's proposition of this third mode of inference led to strong criticism from many philosophers who consider a mere adoption of hypothesis as not being the process of reasoning at all, while others perceive abductive reasoning to be a species of inductive reasoning, and hence, should not be accepted as a distinctive mode of inference (Fann, 1970; Hoffmann, 1999). Kapitan (1992) also argues the autonomy of abductive reasoning on the grounds that the initial conception of a novel hypothesis is not the product of inferential transition, and every inferential phase of the abductive process can be analysed in terms of inductive and deductive methods. Abductive reasoning, therefore, cannot lead to the inference of the best explanation.

In fact, Peirce acknowledges the weakness in validity of abductive reasoning. He accepts that in terms of security, abductive reasoning is the lowest among the three modes of inference. Nevertheless, he asserts that abductive reasoning provides the highest uberty, copiousness or fruitfulness of the three. He claims that abductive reasoning is the fundamental process of scientific knowledge acquisition. It is the only logical operation which introduces any new ideas through hypothesis. He values human imagination as the source of such hypotheses. Abduction is, therefore, the first stage of scientific inquiry where initial hypothesis is invented. Deduction, therefore, later explicates such hypothesis by deducing the necessary consequence which may be tested before induction is used to test such hypothesis (Fann, 1970).

Peirce's sustenance on abductive reasoning underlines its importance as an integral part of the knowledge acquisition process, which needs to be adopted alongside deductive and inductive reasoning. The unique logical form clearly distinguishes itself from other forms of logical inference. Hence, abductive reasoning should be a possible strategy that the learner may use in their reasoning process as well as deductive and inductive reasoning.

With hypothesising a possible explanation as a key property of abductive reasoning, Eco (1983) elaborates abductive reasoning into three different types: 'overcoded abduction', 'undercoded abduction' and 'creative abduction'.

‘Overcoded abduction’ is an abduction where the law is given automatically or semi-automatically, usually based on one’s perception of coded language or plausibility of experience. For example, a postman is usually assumed to be a ‘male officer’ who delivers posts. The hypothesis is placed with a strong degree of confidence as a result of a culturally perceived concept of words or concepts with high probability.

‘Undercoded abduction’ is the abduction where the rule is selected from a series of equiprobable rules but with a criterion based on current world knowledge. For example, when one enters into a language classroom, one usually assumes that the oldest person in the room is the teacher while it is not always the case. But the world experience may draw one to abductively hypothesise that the most senior person is the most likely to be the teacher.

‘Creative abduction’ is an abduction where law must be invented instead of referring to existing principle or knowledge. The case of creative abduction is usually found in a detective novel where a variety of evidence is pieced together into a coherent logical story and multiple stories can be ‘invented’ from the same set of evidence.

This categorisation of abductive reasoning shows varying levels of confidence and plausibility of this type of reasoning. It also illustrates the innate selective guessing ability of humans, which plays a significant role in finding new knowledge. The criterion Eco uses to separate ‘overcoded abduction’ from ‘undercoded abduction’ is the level of confidence, judging from the awareness of other possibilities from the preferred hypothesis. For ‘overcoded abduction’, the reasoner is certain of the hypothesis, since he/she is unaware of any other possible explanation, or has already ruled out other less likely possibilities. For ‘undercoded abduction’, the most likely hypothesis is selected on the basis of a rational criterion but the reasoner is still aware of other equal possibilities. 'Creative abduction' takes place when the reasoner cannot find any known rule to explain the phenomena and has to devise a new rule for the circumstance. This classification is therefore culturally and experiential based rather than logical, and it reflects the affective dimension of the reasoner rather than how they logically adopt abductive reasoning.

From these descriptions of three types of reasoning, it can be seen that each type of reasoning has its own pros and cons and none of them has a logical rigour to ultimately verify human knowledge. Deductive reasoning has the problem of 'posterior analytics', where the generally accepted first premises can always be disputed or challenged. Inductive reasoning relies on possibilities rather than explanation, while abductive reasoning is simply a process of hypothesising without logical verification. These three types of reasoning are, therefore, usually adopted in combination in order to strengthen the validity of a particular knowledge though their inherent weaknesses still hinder the human achievement of finding the ultimate truth.

2.3 ARGUMENTATION

Besides reasoning, another cognitive activity that aims to justify truth is called 'argumentation'. Though the definition of the term 'argumentation' may be diverse among scholars, its central purpose to verify the truth is shared by the process of reasoning. This section discusses different concepts of argumentation especially when it concerns the process of truth verification among individuals.

2.3.1 Toulmin's Pattern of Argumentation

Toulmin (2003) proposes a basic layout of what he calls 'argumentation' in order to verify truth by distinguishing the claim (C), the data (D) and the warrant (W). The claim (C) is the statement to be justified. The data (D) is the evidence used to support the claim. While the warrant (W) is the process of justification or inference rule which connects the data (D) to the claim (C). He outlines the relationship between these three basic entities as shown in Figure 2.2.

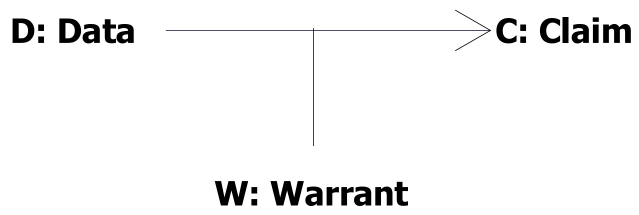


Figure 2.2 Toulmin's pattern of argumentation

Though Toulmin calls the process of justification of rules or principles an 'argumentation', he never made clear what he means by 'argumentation'. Pedemonte, who adopts Toulmin's model of argumentation in her research (2007) about the relationship between argumentation and proof, defines argumentation by its key characteristics. She interprets it as a rational justification used to convince a universal audience'. The term 'universal audience' suggests that argumentation is not just an individual process but also has a social dimension. The acceptance of argumentation should be mutual between one another. Though it is still questionable how Toulmin's proposed layout of argumentation reflects this social dimension, his distinction of claim (C), data (D) and warrant (W) can still be helpful in analysing an individual's reasoning process.

The distinction of warrant is useful to examine different ways one can reason in the argumentation process. Nevertheless, the choice of the term 'warrant' strongly implies 'validity', while in fact one may complete the reasoning process structurally with an invalid result. The use of the term 'warrant' in this research, therefore, would not automatically imply the 'validity' of the reasoning process, but would be considered as the 'way' reasoning is used in argumentation.

Besides proposing this basic layout of argumentation, Toulmin (ibid.) also refines this model by distinguishing further elements: Qualifier (Q) and Rebuttal (R). In order to reflect the subtlety of the warrant process, especially when it does not always fully guarantee the truth, Toulmin places Qualifier (Q) before the Claim (C) notifying the degree of confidence that such 'warrant' can validate the 'claim'. He, then, adds another element Rebuttal (R) connecting with the Qualifier (Q) in order to indicate the circumstance in which the authority of the warrant needs to be set aside. The enhanced model is presented in Figure 2.3 below.

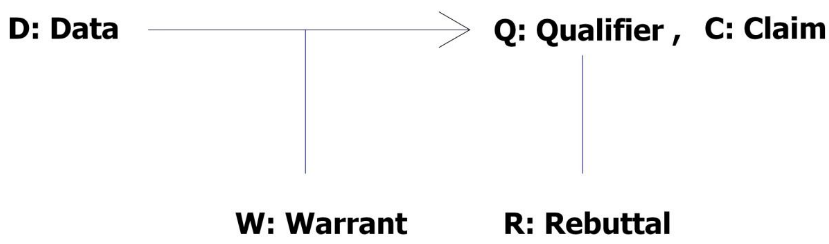


Figure 2.3 Toulmin's enhanced model of argumentation

The 'qualifier', therefore, signifies how strongly the claim can be verified and 'rebuttal' provides the case where the warrant can be negated. Researchers such as Inglis, Mejia-Ramos et al. (2007) advocate this enhanced Toulmin model, claiming that it better reflects the justification process of the subject students in their research. However, in my opinion, the 'qualifier' simply indicates that the ‘warrant’ does not necessarily guarantee the claim, and different degrees of confidence can be involved. There is no indication that this ‘qualifier’ element can be further categorised or systemised to truly reflect the varying nature of warranting the claim. Moreover, the ‘rebuttal’ part can also be viewed as another type of warranting by using a new data to negate the claim. These two additional elements seem redundant, since we can also use the basic model to illustrate the negation of ‘rebuttal’, with the awareness that all these processes of warranting may not be definitive. The main usefulness of Toulmin’s model, therefore, remains its distinction of the ’warrant’ from ‘data’ and ‘claim’, which can give us a better insight into the justification process.

Though Toulmin’s argumentation structure has been adopted by many researchers, Wohlrapp (1987) criticises the model as being inadequate for the wide range of justification processes. He claims that the ‘warrant’ element dogmatises a certain kind of data to claim the transition rule. It ignores alternative ways of justification, especially when a claim is established by mutual agreement or is a constitutional fact. The common circumstance of objection from different parties is also neglected, limiting itself to individual justification. With these weaknesses of ‘warrant’ in Toulmin’s model, Wohlrapp suggests that the general argument should be ‘reasoning’ rather than ‘warrant’ and in the event there are questions, the claim can be further justified by a higher level of reasoning. Wohlrapp’s criticism implies that Toulmin’s argumentation model is not applicable in a real life situation, when argumentation is a collective activity and there is more than one way to justify a claim.

A good example of this conflict can be taken from Rodd’s paper about mathematical warrant (2000). In this paper, Rodd clearly distinguishes the term ‘warrant’ from ‘justification’ by defining ‘justification’ as the process of giving logical reasons for the claim, while ‘warrant’ is something that secures knowledge. She emphasises the individuality of 'warrant', i.e. different persons may have different views of how knowledge can be secured. This personal 'warrant' can also differ from generally accepted forms of justification, such as mathematical deductive proof, since there are cases where students know how to perform proof but have no understanding of what it means. Rodd also shows that there are many ways one can 'warrant' mathematical statements. It may be from the process of formal proof, empirical generalisation, intuitive visualisation or even from testimony or authority of a trustworthy other. These different ways of 'warrant' show that Toulmin's model of argumentation may be too simple to cover all possible justifications, especially in a real life situation. It also highlights the individuality of knowledge claim, which can create tension when members of the society may have different views of how truth can be confirmed.

Though 'warrant' in Toulmin's model may not be able to reflect all possible means of justification, its distinction from the data can still be useful in analysing one's justification process. But we should also be aware that there can be multiple ways of 'warranting' and different people can have their own preference of the knowledge confirmation process.

2.3.2 Other Concepts of Argumentation

So far, the discussion of argumentation focuses on the cognitive level of an individual with little regard to social impact. Baker (2009) concentrates more on argumentative interaction between two interlocutors, in order to study the argumentative pattern during the collaborative problem solving process. He first distinguishes intrapersonal conflict: a conflict within an interlocutor, from interpersonal conflict: a conflict between interlocutors. In the case of intrapersonal conflict, another interlocutor plays the role of helper to the conflicted interlocutor to help him/her decide. He then elaborates the 'conflict' by showing that it can be a conflict between two different possible theses (T_1 and T_2), of which only one can be accepted, or a conflict of whether or not to accept a particular thesis. With these combinations, the four possible argumentative situations can be shown in Figure 2.4.

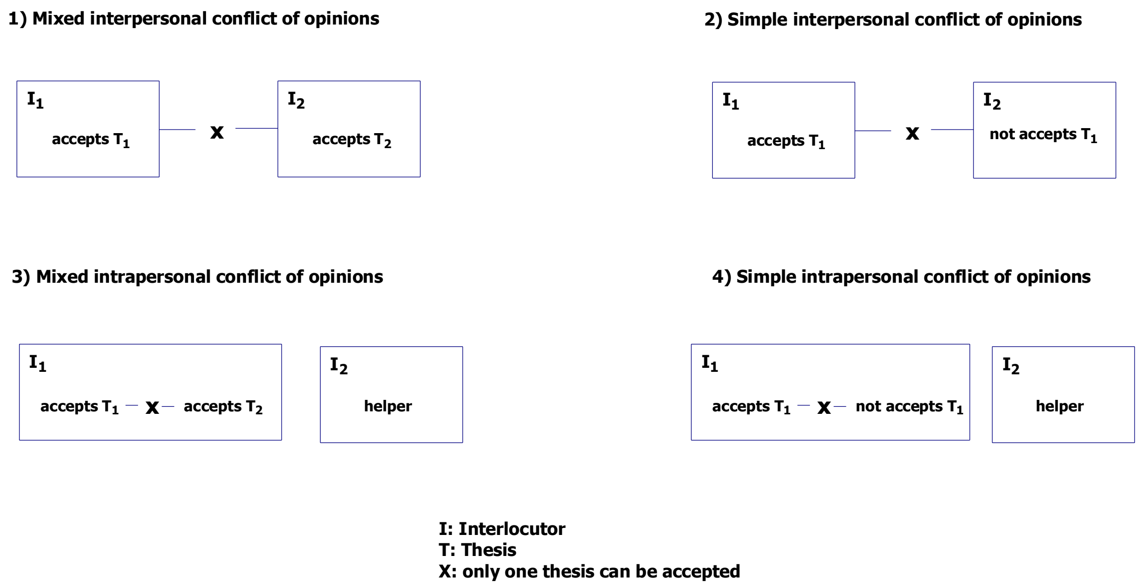


Figure 2.4 Baker's model of argumentative situations

Baker clearly identifies the role of each interlocutor in all these four argumentative situations, even in the case of intrapersonal conflict. Nevertheless, in my opinion, the model seems to be too simplified, especially when the conflict is limited to the either/or case that only one thesis can be accepted. In fact, two different theses can be equally applicable in solving the problem and sometimes multiple theses are needed in order to reach a solution to the problem. These models, therefore, are not sufficient to reflect the nuance of argumentation which can be much more complex. Moreover, the nature of 'conflict' and the role of the 'helper' in intrapersonal conflict cases have not been clarified. Baker's models may not be sufficiently refined to cover all possible argumentative situations.

Mason, Burton et al.'s (2010) view of argumentation may be much less ambitious than Baker's. They suggest that the process of argumentation should involve three different levels, i.e. convincing oneself, convincing a friend, and then convincing the enemy or the sceptic. Besides arguing with oneself to believe in something, Mason, Burton et al. claim that it is also helpful to articulate and externalise one's argumentation to another. This can help one to reflect one's own thought and ensure that such argumentation is logical enough to be socially accepted. The next level is to convince the enemy. This will help the arguer to be even more rigorous about his/her argumentation under the challenge of the enemy. Besides trying to convince an external enemy, Mason, Burton et al. also suggest that one can also play the role of one's own enemy and use such internal enemy to refine one's argumentation. These levels of argument by Mason, Burton et al., seems straightforward, and is helpful to depict how argumentation can be developed in a social setting.

From the varying ideas of 'argumentation' and 'warrant' discussed above, especially how they relate or overlap with the process of reasoning, I would interpret 'argumentation' as a process of truth verification in a social setting, i.e. between two or more interlocutors, in order to distinguish it from individual reasoning. This would involve possible conflict among each interlocutor as depicted in Figure 2.4, though the idea that only one thesis can be accepted would not be adopted. The structure of argumentation of Mason, Burton et al. (2010) would also be considered when the role of each interlocutor can enhance the process of justification. For the term ‘warrant’, I would adopt Rodd (2000)‘s definition as something that is used to secure knowledge, which gives the term the most flexibility since anything can be used as a ‘warrant’ to verify the claim. The identification of ‘data’, ‘warrant’ and ‘claim’ in Toulmin’s model is integrated with the three types of reasoning, placing ‘reasoning’ as the central process of what and how ‘data’ and ‘warrant’ are used to produce a ‘claim’, or vice versa as shown in Error! Reference source not found.. This summarised model should, therefore, provide a useful framework for later analysis of the learner’s reasoning strategies.

2.4 REASONING AND GEOMETRY EDUCATION

This section surveys the applications of different types of reasoning, i.e. deductive reasoning, inductive reasoning and abductive reasoning in geometry education as well as the problem-solving process. It will also introduce a special kind of reasoning used in geometry education, i.e. ‘transformational reasoning’ which can be an alternative way the learner can use to verify the claim. The section will conclude with a discussion of van Hiele’s levels of geometric thinking, which also relates to the process of reasoning.

2.4.1 ‘Proof’ and Deductive Reasoning in Geometry Education

Deductive reasoning plays a vital role in geometry education, especially in Euclidean geometry, which is included in most school mathematics curricula. Euclidean geometry is generally quoted as a prime example of the deductive reasoning process. It is considered a comprehensive deductive and logical system, based on definitions and a set of axioms or postulates: propositions which are self-evident and need no proof or demonstration, and can be treated as premises. Further propositions, then, are subsequently deducted from those definitions and axioms which can be considered in the conclusion (Gray, 1989; Greenberg, 1993; Katz, 1998; Merzbach & Boyer, 1991; Silvester, 2001). This process of verifying a new proposition based on axioms or verified propositions is called ‘proof’ or ‘formal proof’. Euclidean geometry, therefore, provides an ideal platform for the student to learn deductive reasoning from the proof activities.

Despite a seemingly uniform understanding of the term ‘mathematical proof’ among mathematicians, Balacheff (2008) claims that the interpretation of this term in mathematics educational research actually varies. Drawing from various important research papers in this field, Balacheff highlights five different epistemologies of mathematical proof:

1) Proof as a unique contribution of mathematics.

2) Proof as an idiosyncratic activity.

3) Proof as the heart of mathematical thinking and deductive reasoning.

4) Proof as the tool for promoting mathematical communication.

5) Proof as the foundational unity of theorem.

However, a close examination of these five different epistemologies shows that they are not really contradicting each other as claimed by Balacheff, but they are various reflections of the mathematical proof from different points of view. One of the most interesting epistemologies identified by Balacheff in this paper is the idea of proof as an idiosyncratic process proposed by

Harel and Sowder (1998), which raises the fact that students can interpret the word 'proof' completely differently from the traditional definition in the mathematical domain. Harel and Sowder claim that a person's proof scheme consists of what constitutes ascertaining and persuasion for that person, which are entirely subjective and can vary from person to person. Firstly, it is therefore essential for educators to examine how students perceive the term 'proof', which they can easily confuse with everyday life meaning before we can direct them towards the understanding of formal proof in the mathematical domain. This reaction is similar to the issue of subjective notions of 'warrant' in Rodd's paper (2000) mentioned in Sub-section 2.3.1, reinforcing the individuality of the process of truth confirmation.

The subjective understanding of the proof scheme can also lead to another issue where the student can use deductive reasoning to perform the proof process in a partial manner. For a formal mathematical proof to be complete, the deduction should lead the given statement furthest back to the set of agreed mathematical definitions or axioms. However, it is possible that students may deduce the given statement back to a certain extent, and stop where they believe such a statement to be already an absolute rule (although it can be further deduced). It is, therefore, still arguable whether such an incomplete process is accepted as formal proof.

2.4.2 Inductive Reasoning in Geometry Education

Deductive proof is not the only means of ascertaining geometric theorem or proposition in geometry education. Geometric conjecture can also be verified by induction from various examples as elements of conviction. Klauer (1996) proposes that inductive reasoning in geometry can be classified into two main strands by identifying whether the attributes of objects or relationships between objects are to be compared. In the case where the object's attributes are to be compared, similar attributes will lead to generalisation and different attributes will lead to discrimination. Combining generalisation and discrimination then leads to the process of the cross classification of objects. An example of this strand of inductive reasoning is a comparison of geometric shapes with attributes such as the degree of dimension: 1-D, 2-D or 3-D shapes, type of polygons: triangle, quadrilateral, pentagon or regularity or irregularity of shapes.

In the case of relationship comparison, students are supposed to spot the similarity or difference in relationship to identify whether there is a commonality between relationships among objects which can finally lead to system construction. An example of this kind of inductive reasoning activity is an investigation of the intersection angles of a quadrilateral's diagonals; where subjects are supposed to spot that only a certain type of quadrilateral will have its diagonals always perpendicularly intersected with each other, i.e. a square, a rhombus and a kite. Figure 2.5 shows the diagram of these two strands of inductive reasoning.

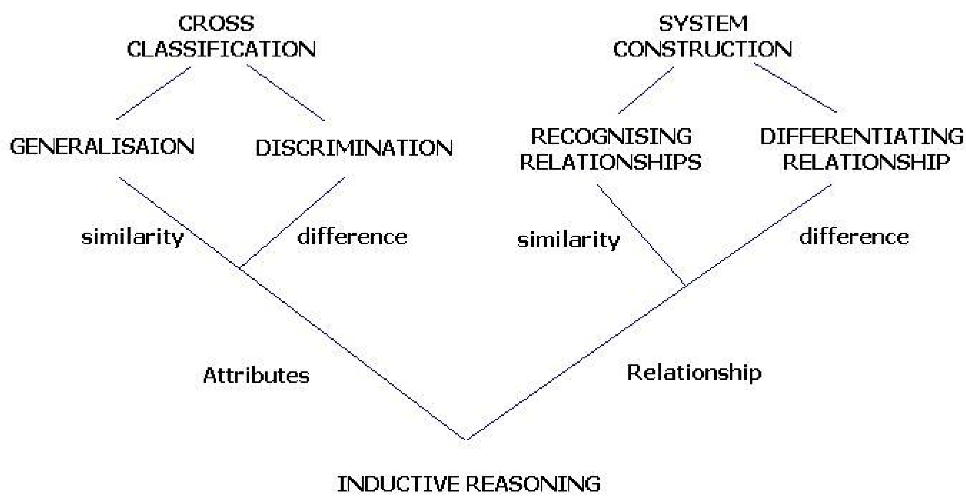


Figure 2.5 Genealogy of tasks in inductive reasoning (Klauer, 1996)

This diagram highlights the comparison process by detecting commonalities which is the core activity in inductive reasoning. The key processes are, therefore, 'generalising' from similarity and 'discriminating' by dissimilarity. These processes can be adopted either to the geometric shapes' attribute or relationship. Note that the higher-level processes of 'cross classification' and 'system construction' involve the process of abstraction from the concrete processes of generalisation, discrimination, recognising and differentiating relationships. This level of inductive reasoning, therefore, implies the ascension from lower-level thinking to higher-level thinking, as discussed in Plato's and Piaget's theories of mental development in Section 2.1, and indicates that inductive reasoning also involves a certain degree of abstract thinking.

2.4.3 Transformational Reasoning in Geometry Education

Besides deductive and inductive reasoning in geometry, Simon (1996) proposes that some students may adopt a different approach in geometric reasoning by inquiring how a mathematical system works within a particular situation. Simon calls this approach 'transformational reasoning' and defines it as:

'the mental or physical enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated'

(Simon, 1996, p. 201)

The key characteristic of transformational reasoning which focuses on the 'dynamic process' of the situation is very much congenial to the dynamic feature in the dynamic geometry environment, to be used in this research where the learner can use drag-mode to explore the 'mathematical system' governing the geometric situation.

Simon (ibid.) illustrates transformational reasoning by showing how a young girl reasons about the relationship between the base angles and the legs of an isosceles. She explains that if the base angles of a triangle are equal, the legs constructed from the ends of the triangle's base will have equal length with a temporal description:

"*if two people walked from the ends of this side at equal angles towards each other, when they meet, they would have walked the same distance*"

(Simon, 1996, p. 199)

This student's perception of an isosceles is not just a static visualisation of the shape but a process and a product of tracing the end points of the triangle's base with an equal angle in the opposite direction (with an assumption of the same speed of both end points). Simon claims that this way of reasoning contrasts with inductive reasoning, where a conclusion is drawn from an accumulation of end-products, i.e. various constructed isosceles. It also contrasts with deductive reasoning, where the validity of the statement is expected to refer to known geometric properties such as congruent triangles.

Nevertheless, the principle of validation in Simon's transformational reasoning still refers to the collection of evidence similar to the case of inductive reasoning, the only difference being continuous dynamic thinking rather than an inference from a collection of static evidence. It is still arguable whether transformational reasoning actually presents a different type of reasoning. It seems to be one of the strategies of inductive inference, and the lack of explanation as to why the system works in such a way still calls for deductive reasoning. Regardless of whether it should be categorised as a special type of inductive reasoning, transformational reasoning can still be an alternative way that students may use to verify geometric statements by recognising the rules governing certain mathematical systems applicable to all circumstances.

2.4.4 Abductive Reasoning in Geometry Education

Abductive reasoning is virtually insignificant in the area of mathematical education research, especially in geometry. There is only limited research in this field which places the focus on this mode of inference. Arzarello et al. (1998; 2002) study students' use of drag-mode in the dynamic geometry environment when given open geometric problems. They find that students adopt the abductive reasoning strategy by exploring the diagram and formulating conjecture before trying to validate it. The research shows that different schemes of drag-mode are used in different phases of the problem. A specific scheme called 'dummy locus dragging' or moving a basic point so that the drawing keeps the discovered property has been linked to the abductive phase of constructing a conjecture, especially when it is switched between theory and drawing.

A recent study by Leung (2012) also illustrates the possible relationship between the learners' dragging scheme and their reasoning process in the DGS environment where maintaining dragging, i.e. dragging that attempts to keep the desired property invariant may lead to geometric abductive reasoning. A task where learners are supposed to study the path of one of a trapezoid's vertices to keep the perpendicular bisectors of the two parallel sides coinciding, Leung demonstrates that maintaining dragging can guide the learners to abductively hypothesise that the traced path of that vertex is a circumference of a circle, with the intersection of the trapezoid's two diagonals as the circle's centre. This example shows how the dynamic feature in the DGS environment lends itself to the learners' process of verification by first making a conjecture prior to a robust construction of such a circle to confirm the conjecture at a later stage.

Other research by Pedemonte (2003) focuses on how abductive reasoning supports the proof process in open geometric problems. Pedemonte's research was also conducted in the dynamic geometry environment. The result shows that when students observe a certain pattern from the diagram, they start with abductive reasoning to formulate the hypothesis of such a case. This stage of abductive reasoning helps them to see the gap between the hypothesis and the verification process, when they feel the need to explain why things happen in such a way, though they may not always be successful at the deductive reasoning stage. Research by Reid (2003), on the other hand, presents an even more rigid interpretation of abductive reasoning. While guiding a class of students through proof of the Pythagorean theorem with the diagram shown in Figure 2.6 where ABCD is known to be a rhombus, the teacher asks the students to prove that ABCD is also a square and encourages the students to conjecture the necessary property to make the rhombus ABCD a square. However, this approach of reasoning strongly relies upon the deduction of the property of a square, and leaves room for the students to choose the property they desire. They may refer to the square's properties of four equal sides and four right angles or its unique diagonal properties, etc. This freedom, therefore, distinguishes selective abductive reasoning from deductive reasoning, where a specific rule or property is explicitly intended. The distinction between these reasoning types is further elaborated in the next sub-section.

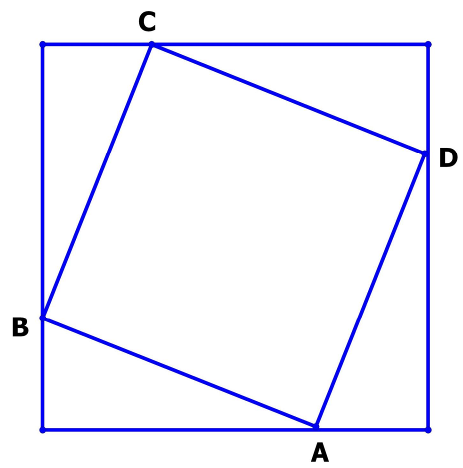


Figure 2.6 Square-rhombus abduction

These studies, although providing just a glimpse of how the student adopts abductive reasoning strategy in open geometric problems, still illustrates the significant role digital technology can play in mediating the reasoning process. It would be interesting to pursue in depth how the dynamic geometry environment relates to the learner's adoption of abductive reasoning strategy, which is not really possible in a traditional paper-and-pencil environment.

Apart from these studies about abductive reasoning in geometric tasks, Balacheff (1988) also introduces a geometric reasoning process called ‘Empirical Justification’. Balacheff classifies empirical justification into three levels, increasing in their reliabilities. They are:

1) ‘Naive Empiricism’, which means an assertion of the truth after randomly verifying only a few cases. This rudimentary verification is considered to be one of the first forms of the process of generalisation.

2) ‘Crucial Experiment’ refers to an experiment whose outcome allows a choice to be made between two hypotheses. The empirical justification at this level is, therefore, based on results from carefully selected examples.

3) ‘Generic Example’ involves making explicit the reasons for the truth of an assertion by means of operation or transformation of an object that is a characteristic representative of its class. Again, the conclusion is based on results from carefully selected examples but with a different purpose.

Balacheff *(*ibid.*)* also distinguishes ‘Generic Example’ from the first two levels of empirical justification by indicating that its strategy no longer refers to the act of ‘showing’ that something is true because it really works, but rather establishes the necessary nature of its truth by giving a reason (ibid.)

Balacheff exemplifies these steps of ‘Empirical Justification’ by students’ strategies to prove the formula, calculating the number of diagonals of a polygon. For ‘naive empiricism’, students gain some confidence in the conjecture from looking at a few cases of polygons. ‘Crucial experiment' takes place when they use a particular case in order to testify whether the conjecture or proposed formula is working or not. The hypothesis used in 'crucial experiment' is set in order to decide whether to accept or reject the proposed formula. This way of hypothesising resembles the structure of 'argumentation' proposed by Baker (2009) on Page 56, where each hypothesis is to be examined to be either accepted or rejected. 'Generic example', on the other hand, involves the examination of a generic case of the polygon in order to find the governing rule explaining the proposed formula. 'Generic example' is, therefore, a significant move from concrete reasoning in the first two strategies, to abstract reasoning where the underlying mechanism is to be sought after.

Though the title 'Empirical Justification' suggests the process of concrete inductive reasoning, its overall process is significantly similar to the process of abductive reasoning discussed in Sub-section 2.2.3. 'Naive empiricism' can be considered as a process of discovering the rule, while 'crucial experiment' may be used as a testing tool at various stages of empirical justification. 'Generic example', can also be mapped to the process of devising an explanatory hypothesis, which is the key action in abductive reasoning. With these similarities, Balacheff's concept of 'empirical justification' can be an illustration of how abductive reasoning may be adopted in the geometric justification process, though he himself does not explicitly relate it to this type of reasoning.

Balacheff also claims that these forms of empirical justification form a hierarchy which students need to go through; from the lower level of 'naive empiricism' through to 'crucial experiment', before they can achieve the skill of the 'generic example' level. This suggests the development stages from lower-level reasoning skills to higher ones by moving from concrete evidence consideration to a more abstract rule as advocated by Plato and Piaget.

2.4.5 Distinction between Inductive, Deductive and Abductive Reasoning in Geometry Education

With the range of ideas and interpretations of inductive, deductive and abductive reasoning in the area of geometry education by different scholars presented in the previous sections, it is useful to draw conclusions from the scope and meaning of each type of reasoning at this stage to provide the basis for this thesis.

For inductive reasoning, Klauer's genealogy of induction (1996) presented in Figure 2.5 appears to clearly identify the two key activities of this type of reasoning: generalisation and discrimination of multiple cases which should also reflect the logical form presented in Sub-section 2.2.2 where a conclusion is made based on the pattern of the premises. The key actions in inductive reasoning, therefore, include generalisation of similar properties in two or more cases and falsification by discriminating one or more cases as counter-example(s). Note that for the process of generalisation, this can be done both from a series of different cases or from a continuous change of the same case, as in transformational reasoning (Simon, 1996) discussed in Sub-section 2.4.3. With generalisation and discrimination as central processes of inductive reasoning, transformational reasoning naturally falls into this category despite the different nature of the characteristics of the cases. Nevertheless, in the case of induction by generalisation, it would account for generalisation of two or more cases, or a case with continuous change only. Generalisation from just a single case would belong to a different inference category and would be called 'naïve empirical inference'.

For deductive reasoning, a confident reference to a known rule or property as a logical explanation of the statement in question is the main criterion used to distinguish it from other types of reasoning. Deductive reasoning can be illustrated with a unique logical form presented in Sub-section 2.2.1 where two premises are used in conjunction to reach the conclusion. However, the essential element of deductive reasoning is the valid logical inference between the premises regardless of the truth. An inference should still be considered deductive reasoning even if the reasoning is false. The following example shows a syllogistic deductive reasoning process where the conclusion is based on a false major premise. The conclusion statement is therefore false even though the reasoning is still deductive.

Major premise: All equilateral triangles are obtuse triangles

Minor premise: ABC is an equilateral triangle

Conclusion: ABC is an obtuse triangle

The main criterion to identify deductive reasoning is, therefore, the attempt by the reasoner to relate the contents of given premises in order to formulate a conclusion. With this principle, the deduction can be valid even when the premises are false.

The most challenging type of reasoning to define is abductive reasoning where different scholars from different periods interpret this approach of inference quite differently. Charles Sanders Peirce proposes a distinct logical form of abductive reasoning as presented in the Sub-section 2.2.3. This form can also be interpreted as deductive reasoning with a fallacy where the logical inference is not valid. The following example shows abductive reasoning as a form of syllogistic deductive reasoning with a fallacy.

Major premise: All equilateral triangles are acute triangles

Minor premise: ABC is an acute triangle

Conclusion: ABC is an equilateral triangle

Despite its clear form, it is also possible that a reasoner intends to adopt this type of reasoning yet he/she is unable to complete the whole process as indicated in the logical form for various reasons. Hence, the proposed logical form of abductive reasoning should not be treated as the sole criterion to identify this type of reasoning. Moreover, scholars may identify the key aspects of abductive reasoning differently, depending on which element is considered most important for abductive reasoning. The issue of how abductive reasoning should be defined is still arguable.

For example, while Charles Sanders Peirce offers a clear logical form of abductive reasoning presented in the Sub-section 2.2.3, Eco gives a broader view of abductive reasoning as the search for a general rule from which a specific case follows (Reid, 2003), emphasising the process of seeking an explanatory rule for the observed circumstance. Pease and Aberdein (2011) also discussed the role of 'surprise' as one possible key element of abductive reasoning, though they eventually concluded that some instances of abductive reasoning may occur without the ‘surprise’ element, such as the abduction in medical diagnosis. On the other hand, Arzarello et al (1998) interpret ‘abduction’ as a shift between the ascending process from the observed case(s) to the tentative theory, and the descending process from the tentative theory to the affirming case(s), reflecting a more thorough process of this type of reasoning, i.e. including both theory formation and confirmation stages. Despite these vivid interpretations of abductive reasoning, it should also be noted that it is not always the case that a reasoner who attempts to adopt abductive reasoning to explain the observed property would be able to complete all the stages as identified by the above scholars. A reasoner may guess that a certain rule is relevant to the observed case yet cannot find the actual relationship between such rule and the case. This unsuccessful attempt should also be counted as abductive reasoning since the overall aim of the reasoner is to find the rule to explain the observed phenomena. For this reason, the fundamental element that should be used to identify abductive reasoning is the formation of an explanatory hypothesis, regardless of the fact that such hypothesis could explain the studied case. This broad definition of abductive reasoning is also adopted by Magnani (2001), Pease and Aberdein (2011), and Leung (2012) and covers a range of reasoning strategies of this type whether it is successful or not. The key difference between abductive reasoning and deductive reasoning in this case is that the premise or rules identified in abductive reasoning is always tentative, while the premise or rule identified in deductive reasoning is selected with the certainty that it will provide a logical explanation of the case.

These identifications of inductive, deductive and abductive reasoning provide a basis for the data analysis stage of this research.

2.4.6 Reasoning and Geometric Problem-Solving

Apart from reasoning, Thailand's B.E. 2544 Basic Education Mathematics Curriculum (IPST, 2001) also identifies problem-solving as one of the higher-order thinking skills that may benefit from the use of digital technology in the classroom. As with reasoning skill, the curriculum does not provide any concrete guidelines on how digital technology may be used to foster problem-solving ability. Though problem-solving is another favoured approach in geometric education in Thailand, it is not the central focus of this research. However, some aspects of problem-solving may also relate to the reasoning process and in this context it is therefore included as a part of this research, but with the main objective being to elicit students' reasoning strategy.

Though the definitions of 'problem-solving' in mathematics education can be diverse, depending on the various perspectives and contexts used, Schoenfeld (1992) summarises the process of problem-solving based on the activity itself as a task to find an unknown means to a focused end. This definition identifies the core concept of the cognitive process of searching for the applicable means to lead to the known end. It indicates a backward approach, where the outcome is clearly identified, challenging the learner to seek a solution to achieve such a goal. Note that this backward process corresponds to the structure of abductive reasoning, where an explaining hypothesis is sought in order to verify the observed fact. Abductive reasoning may therefore be considered a form of problem-solving per se where the goal is to verify the observed fact, requiring an unknown explaining hypothesis as a support.

The mathematics problem-solving process may also require various reasoning strategies in order to help the learner find a valid solution to the problem. Polya (2004) divides the problem-solving process into four stages: 1) Understanding the Problem; 2) Devising a Plan; 3) Carry out the

Plan; and 4) Looking Back. In discussing these four stages, Polya suggests that inductive reasoning can be helpful to spot mathematic or geometric patterns of the situation in the 'Devising a Plan' stage, and the solution verification process at 'Looking Back' clearly needs a process of reasoning. Research by Kolodner, Simpson et al. (1985) also shows that case-based reasoning or reasoning based on analogy to similar case(s) in the past which can be deemed an abductive reasoning, can also be adopted to find a solution in the problem-solving process. Geometric problem-solving tasks therefore can closely involve reasoning strategies and is included as a part of the reasoning tasks in this research.

2.4.7 Van Hiele's Levels of Geometrical Thinking

With the notion that reasoning skill is hierarchical and can be separated into lower and higher-levels, van Hiele, a Dutch mathematics educator, proposes a theory of levels of geometrical thinking, tracing the path through which an individual should go before a higher level of deductive reasoning can be achieved. Based on his experience with students as a school teacher, van Hiele classifies geometric thinking into five levels:

"Level 1: Shapes are recognised as a whole. Student can learn the names of figures.

Level 2: Properties of figures can be identified. Student can describe shape in terms of properties.

Level 3: Relationship among different shapes is established. Student can logically order figures and relationships. Simple deduction can be followed.

Level 4: Deductive reasoning is understood. Student can write formal proof.

Level 5: A systematic development of geometry is appreciated. Different geometry such as non-Euclidean geometry can be understood."

(Usiskin, 1982, p. 4)

These levels are arranged hierarchically where the student must go through the level in order, i.e. the performance in the lower level is the pre-requisite for the next higher level. This model provides guidance for mathematics teachers by identifying what is needed to lead students to the desired higher level.

Note that van Hiele's levels of geometrical thinking shares similar concepts with Klauer's classification of geometric inductive reasoning shown on Page 61. Van Hiele's first two levels can be viewed as a case of generalising and discriminating geometric shapes' attributes in order to perform cross classification. Alternatively, the third level of geometric thinking can be viewed as a case of generalising and discriminating geometric relationships in order to provide system construction.

Though van Hiele's model shares certain characteristics with Piaget's Stages of Cognitive Development, especially in terms of hierarchical order and identification of each level by operations, van Hiele's belief in the human cognitive process radically differs from Piaget's. While Piaget regards the transition from lower to higher levels as biological development which cannot be stimulated by a learning process, van Hiele proposed that the development can be encouraged by careful guidance from the teacher. Van Hiele also emphasises language as a salient medium for thinking: “*Without language, thinking is impossible*” (van Hiele, 1986, p. 9). He even places language as one of the key properties of his proposed levels of geometrical thinking by stating that each level has its own linguistic symbols, and its network of relationships connecting those symbols (Usiskin, 1982). This property reminds teachers to use appropriate language for each level when they communicate with their students. He also criticises Piaget for his negligence of the importance of language in a child’s development (ibid.). Van Hiele’s view on cognitive development theory, therefore, closely relates to Vygotsky’s socio-cultural perspective, especially on the notion that language is an indispensible mediating tool for higher-mental functioning, though it is unclear whether van Hiele was aware of Vygotsky’s works.

After van Hiele’s theory of levels of geometric thinking was introduced in the United States in the 1970s, much research has been conducted to investigate its validity. Studies by Usiskin (1982), Mayberry (1983) and Burger and Shaughnessy (1986) confirm the hierarchy and identification of levels, i.e. students at a certain level can generally perform the operations in lower levels but not those at higher levels. Fuys, Geddes et al. (1988) also confirm the possibility of guiding students to a higher level through instruction. However, Gutiérrez, Jaime et al. (1991) point out that though most students show a dominant level of thinking, their response to geometric tasks usually reflect the presence of skills at other levels. Some students may even show two consecutive dominant levels simultaneously suggesting that they are in transition between levels. So, practically, van Hiele’s theory of levels of geometric thinking should not be approached as discrete divisions where each student can be assigned to a single level at a point of time. Gutiérrez, Jaime et al. (ibid.) even propose that three different degrees of acquisition, i.e. ‘low acquisition’, ‘intermediate acquisition’ and ‘high acquisition’ also exist as transitions between two adjacent levels.

Comparing van Hiele's level of geometric thinking to Piaget's stage of cognitive development, it can be inferred that van Hiele's Level 3 is a transition from the Concrete-Operational to Formal-Operational stages in Piaget's theory, where van Hiele's Level 4 should be mapped to the Formal-Operational stages. The threshold distinguishing higher-order thinking from lower-order thinking in van Hiele's model should lie between level 3 and 4.

Despite general acceptance of the validity of the original levels of geometric thinking, van Hiele later modifies his own model by condensing the hierarchy into three levels as follows:

"1) 'Visual Level'; where figures are judged by their appearance.

2) 'Descriptive Level'; where figures are judged by their properties.

3) 'Deduction Level'; where properties are logically ordered and deduced from one another."

(van Hiele, 1999, p. 311)

The rationale for this modification is not made clear by van Hiele though it is obvious that the 'Deduction Level' covers levels 3-5 in the original model. The threshold for higher-order thinking in this modified model should, therefore, lie between the 'Descriptive Level' and 'Deduction Level'.

From the above discussion of reasoning in geometry education, it can be seen that all major types of reasoning, i.e. inductive reasoning, deductive reasoning, and abductive reasoning can be adopted by students to solve geometric tasks or verify a geometric hypothesis. A special type of reasoning, 'transformational reasoning', is also discovered as an alternative method the student may use to explain some geometric situations. The hierarchy of reasoning skills is also identified and discussed by Balacheff and van Hiele, indicating the trace of development of justification strategies, from the most rudimentary form to the higher-level of abstract reasoning.

The details of these types of reasoning and how they relate to geometry education discussed in this section will provide the basis of information for an analysis of the student's reasoning skill during the empirical stage of this research, especially when characteristics of the student's thinking and the transition from lower to higher-order levels are to be examined.

2.5 ROLES OF DIGITAL TECHNOLOGY IN EDUCATION

This section discusses role digital technology may play in an area of education. It starts with a survey of ideas about this relationship, followed by the identification of a particular role being pursued in this research.

Stevenson (2008) identifies metaphorical descriptions of digital technology in a pedagogical context into four main functions. It can be used as a 'tool', 'resource', 'tutor' or as an 'environment'. Digital technology can be described as a 'tool' if it is used as a means to achieve a desired goal such as producing drawings or documents with particular software applications. However, if digital technology is adapted in existing educational practice with a purpose defined by some form of curriculum, its use is generally described as 'resource'. Examples of digital technology as a resource include the teacher's use of an interactive whiteboard or online-learning modules. Digital technology can also be described as a 'tutor' when an expert system is used as a pedagogical medium to coach, instruct, diagnose, practice or scaffold learning through direct interaction between learner and computer. Most tutorial packages available in the market belong to this description. The final metaphorical description is digital technology as an ‘environment’. In this circumstance, learners are actively engaged in building their own meaning as they work with digital technologies. They control their own trajectory through exploration, experiment and personal creativity in a new milieu. The concept of ‘micro-world’ as small, simple slices of reality where learners build their own understanding illustrates this description of digital technology as an environment (ibid.).

It should be noted that these four descriptions are not necessarily mutually exclusive since certain description can also be integrated into other descriptions. For example, the teacher can use a ‘tutor’ system in their technological ‘resource’ of online-learning, or certain software features can be used as an exploration ‘tool’ in the technological ‘environment’ of such software. They just highlight the different roles digital technology can play in an area of education.

Using digital technology as a ‘resource’ and a ‘tutor’ clearly suggests an enhancement of teaching and learning activities in terms of efficiency over non-technological circumstances. While using digital technology as a ‘tool’ and an ‘environment’ provides novel and distinctive ways of learning totally different from traditional approach. In direct interaction between learners and technology, a digital ‘tool’ and an ‘environment’ can actually assist in mediating the learning process. It therefore plays an essential role in shaping the knowledge gained from the learner’s experience with the ‘tool’ and ‘environment’.

Nevertheless, many teachers value digital technology according to its efficiency rather than the possibility of providing innovative learning. A study on teachers’ representations on the successful use of digital technology in classrooms by Ruthven, Hennessy et al. (2004) shows that technology was perceived as an aid to improve productivity; to support checking, trialling and refinement of data; to enhance the variety and appeal of classroom activities; to promote students' independence and peer support; to overcome difficulties and build the learner's confidence; to broaden reference and increase currency of activity; and to focus attention on overarching issues. None of these suggests the use of digital technology as a mediator for learning, though it is apparent that there is software which can be used to provide such a technological 'environment'.

In the area of mathematical education, Noss and Hoyles (1996) survey and categorise the mathematical software available in the market into two broad types. The first type is software which is designed to repackage existing content knowledge simply for educational consumption, as can be found in various tutorial packages. Another type is application software, which provides learners with a mathematical platform where they can experiment with their mathematical ideas. This helps them to gain feedback which can shape their understanding of mathematics. This latter type of software illustrates the use of digital technology as a 'tool' and an 'environment', as described by Stevenson (2008). Examples of this type of software include LOGO, spreadsheet, Computer Algebra System (CAS) and Dynamic Geometry System (DGS).

From the above discussion, one should be able to conclude that the most unique role digital technology can have in education is as an 'environment'. Using digital technology as an 'environment' brings about a novel circumstance fundamentally different from other traditional settings. This role of digital technology will therefore be pursued in this research since it reflects the most innovative use of technology in contemporary pedagogy.

A notable group of studies on technology as an 'environment' can be found in the case of LOGO. LOGO is a programming language developed in the late 1960s by Seymour Papert and his team (Ross, 1983). Its most widely used feature is ‘Turtle Graphics’. ‘Turtle Graphics’ is a vector graphic programming language which uses text commands such as; FORWARD, BACKWARD, RIGHT TURN, LEFT TURN, with numbers/variables to control a metaphoric turtle’s movement as displayed from the top-view of a 2-D plane on screen. Using Turtle Graphics to design desired traces of the turtle enhances learners’ spatial intelligence when they have to relate the turtle’s movement to the available commands by imagining themselves as a turtle. It can also help learners to acquire mathematical skills such as a sense of angle sizes and use of variables which provide a useful foundation in school mathematics (Papert, 1980).

Hoyles and Sutherland (1992) conducted a three-year longitudinal research on the use of Turtle Graphics in a secondary school mathematics classroom in order to examine how Turtle Graphics might provide an environment for the experiential learning of mathematics. The result shows that the Turtle Graphics environment seems to enable students to reflect on their actions in direct mode, and then capture them in symbolic form, bridging their actions and their generalisation. It also fosters students’ understanding of various mathematical concepts such as decimals, ration, proportion or variable since they can concretely witness these concepts’ practical uses within the activity. Working with Turtle Graphics can encourage students to gain the useful concept of sub-procedure where several commands are stored into a sequential manner especially in the process of debugging. A similar study by Noss and Hoyles (1996) also shows that commands in Turtle Graphics can provide situations where learners come to realise the relationship between negative number inputs and the vertical reflections of the image. Research by Weir (1987) also shows that learners with learning difficulties such as dyslexia and autism may find Turtle Graphics a better platform for learning the mathematics concept. It is found that Turtle Graphics allows a child who possesses a stronger visual-pictorial mathematics skill over verbal-logical skill to solve problems he/she cannot do in a non-visual environment.

These research results clearly illustrate the relationship between the distinctive features in Turtle Graphics environment and the learner's shaping of mathematical knowledge, which is unlikely to take place in a non-technological environment. Besides the influences Turtle Graphics has on the learner's learning process, Papert (1980) also suggests that working with Turtle Graphics provides learners with an opportunity to learn mathematic principles in a natural way similar to the situation of learning a foreign language by actually living in that foreign country. Moreover, its visual display can help concretise abstract concepts facilitating young learners' mental development to acquire a higher level of thinking. This brings about a general notion that a technological environment can also play another role in cultivating the learner's higher-order thinking.

This idea conforms to the view of Vygotsky's socio-cultural theory of cognitive development where an individual's higher mental functioning originates from actions on the social level. It is then internalised into thought on a psychological level, mediated through a cultural symbol or sign system, primarily language (Vygotsky, 1978). Vygotsky calls the difference between the level of a child's actual development as determined by independent problem solving, and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers as 'Zone of Proximal Development' or ZDP *(*ibid.*)*. This concept emphasises social interaction in a child's mental operation. The model places social interaction as a primary factor of an individual's development of higher mental thinking. Although Vygotsky's original concept of ZDP illustrates social contact mainly as human interaction (adult, more capable peers), Ravenscroft (2001) suggests that this social contact may be extended to an intelligent computer system where the learner can also engage in dialogue in order to learn new concepts.

This section shows how digital technology can be useful in education in a number of ways. However, the most distinctive and exclusive benefit digital technology may play in education is the provision of a novel 'environment' where students can learn from their experience. The case of using LOGO to help students learn mathematics illustrates this benefit and suggests that a technological environment may help enhance students' thinking skill. This use of technology as a learning environment will be the setting of this research in order to examine how it relates to the students' development of reasoning skill.

2.6 DYNAMIC GEOMETRY SYSTEM AND ITS FEATURES

This section provides an overview of Dynamic Geometry System (DGS). It discusses the definition of the DGS based on compulsory functions, a comparison of a range of available DGS products and some relevant issues of the way DGS portrays Euclidean geometry.

2.6.1 Definition of Dynamic Geometry System (DGS)

Though the term Dynamic Geometry System (DGS) has been used by mathematics educators for decades, the standardised definition of the term has not been established. Hölzl (1996) summarises three common features of DGS. It should be able to:

" - simulate ruler and compass constructions as laid down in Euclid's Elements

- support those constructions by macro that can be defined by the user

- allow certain parts of a figure to be moved without changing its underlying geometric relationships."

(Hölzl, 1996, p. 169)

While Goldenberg and Cuoco (1998) give a broad description of DGS as:

"The software (which) provides certain primitive objects e.g. points, line, circles / basic tools e.g. perpendicular to a line or through a point for assembling these into composite objects, and several possible transformations . . . It also allows the user to measure certain parts of the drawing and typically to trace the path of points, segments, or circles as dynamic transformations are applied.

"The feature that distinguishes DGS from other geometry software (is) the continuous real-time transformation often called 'dragging'. This feature allows users, after a construction is made, to move certain elements of drawing freely and to observe other elements respond dynamically to the altered conditions. As these elements are moved smoothly over the continuous domain in which they exist, the software maintains all relationships that were specified as

essential constraints of the original construction, and all relationships that are mathematical consequence of these"

(Goldenberg & Cuoco, 1998, pp. 350-351)

Sträßer (2002) refers to a German description by Grauman, Hölzl et al. (1996) from *Journal fur Mathematikdidaktik* which defines three common features of DGS as:

"- a dynamic model of Euclidean school geometry and its tools (the dragmode)

- the ability to group a sequence of construction commands into a new command (macro-constructions)

- the visualisation of the trace of points which move depending on the movement of other points (locus of points)"

(Sträßer, 2002, p. 65)

The way these definitions are presented seems to be more as a description of what DGS can generally do rather than an explicit identification of what exactly are the essential components in order to be called the software Dynamic Geometry System. However, the descriptions by Hölzl and Goldenberg and Cuoco give a hint that dragging, or the fact that figures can be moved is the most important feature in dynamic geometry software, distinguishing it from other graphical software. It also belongs to one of the common features in the German definition referred to by

Sträßer, though they refer to it as drag-mode. It should be reasonable to conclude that dragging or drag-mode is one of the essential features in a dynamic geometry system.

In fact the term 'drag' has already been mentioned by many scholars even before the term 'dynamic geometry software' came to be known. The word is used when they discuss certain geometry software such as Cabri-Geometre or The Geometer's Sketchpad. Laborde (1993, p. 56) characterised the drag feature in Cabri-Geometre as '*A drawing can be continuously modified while preserving its description when part of its independent elements is moved*'. Hoyles and Noss (1994, p. 716) give a clearer explanation of 'dragging' as '*moving one of the basic components of the construction about the screen by the mouse, the relationships among those points, lines, and circles are preserved*'. All these and other similar descriptions of dragging emphasise the retention of geometric relationships either pre-defined or theoretical among the drawing or figure's components, while a certain part of such drawing or figure is moved (Balacheff, 1993; Bloomfield, 1992; Green, 1992; Schumann, 1992).

Drawing from these descriptions of DGS and drag-mode, I would summarise the fundamental elements of DGS as follows:

1) DGS should allow the user to create basic geometric objects such as points, lines, segments or circles on screen.
2) DGS should allow the user to construct a new object with a pre-defined relationship to existing object(s) such as a parallel or perpendicular line, midpoint, bisector etc.

3) DGS should contain a drag-mode where a mouse can be used to grasp and move certain parts of the drawing or figure while both the geometrical and pre-defined relationships among elements are retained.

All DGS should therefore posses all these three elements. Other features such as macro (grouping of commands) or locus (tracing the path of an object) will be treated as additional features. With this definition, current examples of DGS include Cabri-Geometre, The Geometer's Sketchpad, Geometry Inventor, Cinderella, GEOLOG and Thales.

2.6.2 The Geometer's Sketchpad and other DGS products

Though The Thai version of The Geometer's Sketchpad (GSP) which is selected as the dynamic geometry environment in this research satisfies all fundamental elements identified in the previous sub-section, its functions may differ in detail from other similar products available in the market. This difference raises the issue of how GSP may represent the DGS as generic software, especially when GSP has its own unique features and characteristics distinctive from the other software in this category.

Comparing GSP with other popular DGS products such as Cabri-Geometry (J.-M. Laborde & Bellemain, 2003), Cinderella (Richter-Gebert, 1999) and GeoGebra (Hohenwarter, 2008), brings about substantial variations of the dynamic geometry environment even though they share the same aim which is to portray Euclidean geometry in a dynamic mode. In terms of origin, the genesis of GSP is the integration of a vector-based 2D graphic drawing tool with drag features found in the illustration programme called MacDraw in Macintosh by Nick Jackiw. It was not intended as 'dynamic geometry software' per se in the first place. The geometric rules were incorporated in the software afterwards (Scher, 2000). This contrasts to Cabri-Geometry where Laborde (2003) originally developed it as an Euclidean platform in a dynamic mode with strong reference to the tradition of the paper-and-pencil environment. Its sketch is even designed as a finite virtual one square metre sheet of paper neglecting the possibility of infinite plane as adopted in other software. The foundation of Cinderella is even more sophisticated. Seeing the inadequacy of basing the software on a plane of Euclidean geometry, Cinderella's developers Richer-Gebert and Kortenkamp (1999) designed the platform on projective and non-Euclidean geometries and treated plane Euclidean geometry as a part or variant of the whole system. This makes Cinderella a unique DGS tool that genuinely includes hyperbolic and elliptic geometries without the need of add-on features as in other products. GeoGebra, on the other hand, has the broader goal of joining Euclidean geometry, algebra and calculus into a single system. Markus and Judith Hohenwarter (2008), the developers of GeoGebra focused more on the relationships between geometry and algebra. They established software with a bidirectional connection between geometry and algebra systems allowing users to manipulate either the diagram or the parameters of the algebraic function and see the relationships. These different approaches to software design bring about a diversity of the DGS products available in the market, even though they all share the same central purpose of portraying Euclidean (or non-Euclidean geometry) in the dynamic mode.

Apart from the varying original rationales of these popular DGS products, the freedom of the designer's choice to programme the software in any particular algorithm can also change the behaviour of different DGS. One of the most significant differences can be found in the interface design of GSP and Cabri Geometry. For the user to execute a particular command from the menu, the correct object or combination of objects needs to be selected before such a command is enabled in GSP. While in Cabri Geometry, the user must select the command from the menu before applying it to the object(s) on screen. Though GSP's algorithm to select object(s) before a command may make sense in terms of software design, Hardy and Wilson (1995) claim that such an algorithm may disrupt the users' thinking process when they need to construct a new object based on certain conditions. Cabri Geometry's command-before-object algorithm, on the other hand, supports the user's mental object process where the desired object is imagined then the appropriate command is located to define the property of such object. This argument does not make much sense since the user needs to think of both the operation and the object to execute at the same time in order to find the solution. The order of object or command selection does not make a significant procedural difference and GSP or Cabri Geometry should be able help the user solve the problem as effectively. The command-before-object or object-before-command algorithm seems to be a matter of user preference and does not genuinely affect the problem solving process. The only problem for object-before-command in GSP lies in the case where the user fails to select a correct set of objects in order to enable the desired command. This can simply cause confusion, especially when GSP does not provide instant feedback to guide the user to the selection, as criticised by Hardy and Wilson (ibid.). Nevertheless, this problem stems from the inadequate interface design and is not the result of the chosen algorithm.

Other differences can be found in how each DGS treats the behaviour of parent-and-child relationships between objects. While most DGS stick to a hard-and-fast rule that the user can move only the parent object and not its child(ren), GSP opts for reversibility and allows the user to drag the child object where possible (Scher, 2000). Such reversibility provides significant flexibility for the user's exploration of the construction since more objects can be moved. For example, a midpoint of a circle's chord can be moved in GSP but not in Cabri Geometry, Cinderella or GeoGebra. GSP's reversibility of the parent-and-child relationship makes it a more powerful exploration tool, though the reversibility remains partial since some of the child objects cannot be moved or dragged, as this would just translate the whole construction around the screen. Such partial reversibility, therefore, may present inconsistency in parent-and-child behaviour in GSP.

These diverse programming approaches and behaviour of different DGS products shatter the generic notion of how the DGS should perform. The freedom of the programming choices of the designer can make one DGS significantly different from another, despite their common goal to portray Euclidean geometry dynamically. Lester (undated) suggests that such inconsistency among DGS products calls for a common formulisation of how Euclidean geometry should be presented in the dynamic mode. The existing axioms are no longer practical or sufficient to define Euclidean geometry in motion.

Choosing GSP Thai Version as a tool to be used in this research, therefore, represents just one possible DGS environment. It cannot be viewed as a DGS in a generic sense since the unique command algorithms embedded in the system would be a significant part of the whole environment. To be precise, this research is conducted in a GSP Thai Version environment and it may lead to a significantly different outcome had this research been conducted with other DGS products. Though GSP is chosen in this research solely because of its availability in a Thai version, it would still be selected had other products been available in Thai, since it provides the learner with the broadest opportunity to explore the geometric construction due to the parent-and-child reversibility.

2.6.3 Issues of the Euclidean Geometry Portrayal in the DGS

Laborde (1993) gives an insightful analysis of the difference between drawings in a paper-and-pencil environment and figures in a dynamic geometry environment. She starts by discussing the dual roles of drawing, either on sand or paper or on a computer screen as a material entity and an object of theory. The material entity accounts for the visible part of the drawing which we can perceive with our eyes, while the object of theory belongs to the world of 'idealities'. This means that the material drawing cannot be completely faithful to the theoretical object. The fact that it is impossible to draw a visible widthless line distinguishes the material entity of a drawing of a line from its theoretical definition. The static drawing in a paper-and-pencil environment also presents other problem. The properties of an object can be unclear in certain cases. To give a very basic example, there is no way to tell whether point C on the circle A in Figure 2.7 is a point on the circumference or a free point unless one can try to move point C or circle A around.

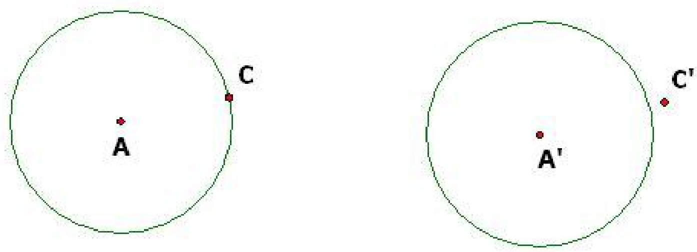


Figure 2.7 C as a point on circumference or a free point?

However, with dynamic geometry software which can keep the properties and relationships between objects of the drawing when part of it is moved with the drag-mode, the drawing now can contain the description. This description will direct the behaviour of the object according to the direction of movement. To make a distinction, Laborde calls a drawing that can retain geometric description in dynamic geometry software as a ‘figure’ and differentiates it from the static ‘drawing’ more accustomed to a paper-and-pencil environment.

Balacheff (1993) raises another issue about certain behaviour in dynamic geometry software which may cause confusion to learners. This issue stems from the decisions of software designers in the course of their programming where they have to characterise the certain behaviour of objects when a certain point is dragged. The basic example is given when one drags an endpoint of a segment with a point on the segment. The designer needs to decide how the point on the segment should behave when one endpoint is dragged (see Figure 2.8).

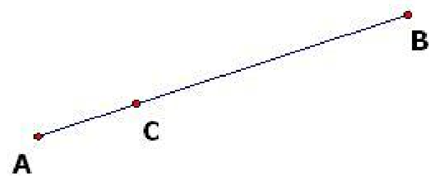


Figure 2.8 How should point C behave when point B is dragged?

In Cabri-Geometre and the Geometer’s Sketchpad, point C will keep the ratio of distance between the two end-points while point B is dragged. But in Euklid, another DGS product, the distance between point A and C will remain constant (Hölzl, 1996). This behaviour is completely dependent on the designer’s decision and has nothing to do with the concept of geometry itself.

However, it can be an interfering factor when students are supposed to learn a new geometric concept with the tool themselves.

Drag-mode brings about a unique feature in DGS. In order to retain the pre-defined relationship while the figure is moved, the software designer needs to incorporate an order or relationship among each element of the figure. This relationship is called 'parent-and-child relationship' denoting the dependency of a new element on the previously constructed element. The element related to a previous element is called a 'child' and the previous element is called its 'parent'. Figure 2.9 illustrates the parent-and-child relationship between vertices and segments of a rectangle. When the vertex B is dragged with a mouse, all other segments will move in a manner that will keep the shape ABCD as a rectangle since each segment has already been defined to be perpendicular to the adjacent sides. These segments, therefore, would be considered children of point B which is their parent.

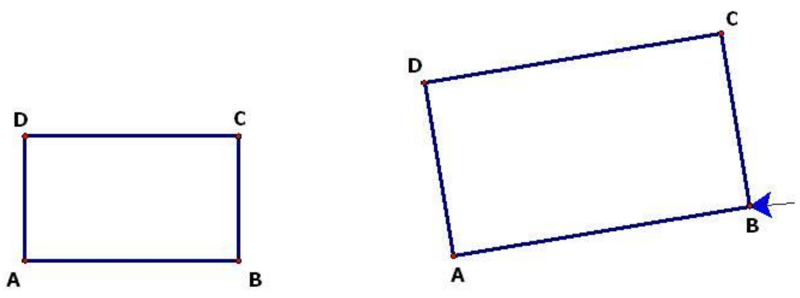


Figure 2.9 Parent-and-child relationship in a rectangle

Hölzl, Healy et al. (1994) and Talmon and Yerushalmy (2004) define parent-and-child relationship as a hierarchy of dependencies between the elements of the construction. The use of the term 'hierarchy' to define a parent-and-child relationship in this definition is arguable since the word ‘hierarchy’ implies vertical levels of importance from the highest to the lowest, while there is no indication that a parent is more important than its child(ren) in a parent-and-child relationship. The relationship rather indicates the dependency of movement between the figure’s elements. As a matter of fact, applying the relationship between ‘parent’ and ‘child’ as a metaphor to such a dynamic relationship can be easily misleading. The term ‘parent-and-child’ can be interpreted in a number of different ways such as:

1) Parent entities produce a child entity which is not the case in DGS since the child is actually produced by the user through commands with a defined relationship to an existing element.

2) Parent entities are always larger than a child entity when compared to the size of adults and their children, which obviously is not the case in DGS environment. It is possible for a child entity to be bigger in size than its parent entity.

3) Parent entities always come before a child; this is partially true in DGS since the parent must be constructed before a child. However it implies no movement in dependency which is the essential meaning of this relationship.

4) Parent entities control the behaviour of a child entity as in the case where in real life parents have authority to groom their children. This metaphoric interpretation can be the most misleading since the behaviour of the child is not under the control of its parent(s). It is actually under the control of the user in the use of drag-mode through the dependent relationships via its parent(s)

Nevertheless, the metaphor of ‘parent-and-child, even though problematic, has become a common reference to this kind of relationship, which reflects only the fact that a child’s movement is dependent on its parent(s).

The issue of software design together with the dependency nature in dynamic geometry software can contribute to the ambiguity of a figure’s properties in the dynamic geometry environment. In such cases different results may be observed when the vertices of a square are dragged. The square may change its size or the size may retain but the whole square is translated around the screen. This can lead to the issue of the symmetry of such a square in the dynamic geometry environment since the points actually have different properties when dragged though they all look exactly the same.

2.6.4 Measurement Feature in the DGS

Besides the drag-mode and parent-and-child relationship, measurement is another feature found in most DGS products where length, distance, angle, area or slope can be displayed in a preferred unit (if applicable). Though authentic Euclidean Geometry never took measurement into account in its deductive inference system, measurements may be used as an alternative means to help affirm conjecture in practice. The measurement feature in DGS also presents a distinctive advantage when all measured values are dynamic, i.e. persistently adjusted when the figure is dragged. This helps the user to notice the variant and invariant properties numerically, in addition to visual evidence. Nevertheless, the dynamic measurement feature in DGS can also be problematic. Olivero and Robutti (2007) point out that measurement in DGS needs to be based on the computer's graphical screen unit of pixel, which significantly differs from the theoretical Euclidean Geometry plane where the number of points is infinite. Measurement in DGS therefore provides only approximation since its precision is limited by the dimensions of the pixels. A more serious problem arises from the fact that the value of these measurements cannot be displayed infinitely onscreen and need to be presented as rounded numbers, based on the number of units preset by the software users. This causes concern in circumstances where measurements displayed in the DGS environment may contradict with the Euclidean geometry theory. Figure 2.10 illustrates a diagram where the displayed measurements in The Geometer's Sketchpad violate the properties of the sum of internal angles of a triangle.

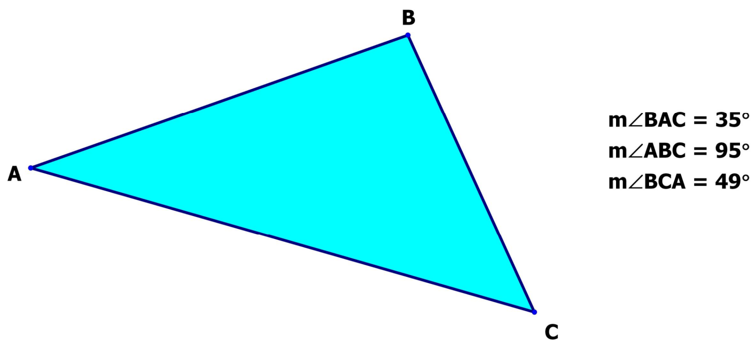


Figure 2.10 Measurement anomalies in DGS

Research by Ruthven, Hennessy et al. (2008) shows that such measurement anomalies in DGS can have a significant effect on classroom teaching practices, especially when a new geometric rule is to be introduced. They found that teachers adopt different strategies to deal with this problem, ranging from deliberately demonstrating the sketches in such a way that measurement anomalies will not occur, to openly explaining the concept of rounding in DGS measurement to students beforehand. Dynamic measure in DGS, therefore, can be both advantageous and disadvantageous when used in geometric exploration tasks.

The discussion of how DGS portrays geometry in the above sub-sections illustrates that the dynamic behaviour provided by this software is not entirely geometrical. They also incorporate the choices made by the designer and necessary rules such as parent-and-child relationship in order for the software to work in a dynamic mode. Using DGS in a learning environment would present both geometric and software design elements to the students at the same time, which is similar to the LOGO case. It generates a crucial issue which needs careful consideration when DGS is to be used in geometry education.

2.7 TASK DESIGNS IN THE DYNAMIC GEOMETRY SYSTEM

With the sheer flexibility of the DGS, it can be integrated into classrooms in a number of ways. The first level of categorisation can be determined by how the DGS is used to support different classroom activities, i.e. how the learning environment is set. The teacher may use the DGS as an illustration tool with overhead projection to lead a whole class discussion or he/she may choose to let the students use the software to learn the geometry concept on their own. However, this research focuses on the learning activity where students have a first-hand experience with the DGS in order to investigate their cognitive process in the environment. These interactions are examined through a set of purposive tasks aimed at challenging them in a range of aspects.

With the hands-on learning activity, where students have an opportunity to interact with the software on their own, different kinds of tasks can be adopted to encourage different skills and means of knowledge acquisition. Note that the word 'learning activity' in this thesis refers to the action students should take in order to learn new concepts, e.g. using the DGS tool to learn geometry themselves, while 'task' refers to a goal-oriented challenge or instruction during such action. This section surveys a range of approaches to task design in the DGS environment from the literature presented in the following seven sub-sections.

2.7.1 DGS Commands Familiarisation Task

The main purpose of this kind of task is to introduce learners to the commands available in the DGS. It consists of a series of disconnected instructions aimed at allowing the learners to familiarise themselves with the software features. This kind of task is adopted by one of the participant teachers who is new to mathematics teaching, but is an expert in technology, in a study about the implementation of the DGS tool by teachers of varying experience and backgrounds by Laborde (2001). Though this kind of task is technologically oriented rather than geometrically, it is still helpful for the teacher or the researcher to see how the students initially react to the new environment and how they regard the benefits of the given software.

2.7.2 DGS Conditions Familiarisation Task

Besides familiarising the students with the available commands in the DGS, some teachers or scholars may also design tasks to let the students experience particular conditions or command algorithms in the DGS environment, especially the parent-and-child relationship. A study by Hölzl, et al. (1994) adopts a simple geometric construction task where students are supposed to construct a robust parallelogram which remains a parallelogram under dragging in order to examine their understanding of the inherent parent-and-child relationship in the DGS construction. Another study by Talmon and Yerushalmy (2004) deliberately designs a set of tasks where students are supposed to construct two right angle triangles with a different order of construction and then predict the triangles' behaviour under dragging to investigate their innate perceptions of the parent-and-child relationship. The task design of these two studies has the specific purpose of eliciting the students' impression of the specific condition of the parent-and-child relationship in the DGS environment, an essential foundation for later interpretations of the figures' dynamic behaviour in this platform. Again, this type of task design focuses more on the technological aspect of the DGS environment rather than geometrical, but still plays a very important role for the students to make sense of the whole dynamic system of the DGS.

2.7.3 Geometric Concept Introduction Task

In Laborde's study (2001) mentioned above, different teachers also design tasks where students are supposed to examine and observe the properties of geometric entities such as a vector or a product of geometric operations such as translation, and then draw a conclusion based on their observations. This kind of task views DGS as a pedagogic tool to introduce new geometric notions to the students and allows them to construct the meaning of such a notion on their own through direct interaction with the object represented by the DGS. Besides the meaning of a new geometric notion, important properties of such a notion can also be studied with the same approach through the task questions. The teacher who designs the task in this study anticipates that students can realise that the magnitude and direction are distinctive properties of a particular vector, while other factors such as the position of the vector plays no part in its property. The role of the students in this kind of task is to conceptualise a geometric notion through observation, with the DGS providing a flexible platform so that students can directly experiment with the studied object.

2.7.4 Heuristic Exploration Task Leading to the Reasoning Process

Another significant set of tasks found in research about DGS use in the mathematics classroom is the heuristic exploration task, where students are encouraged to examine certain geometric situations and are expected to discover interesting geometric property, challenging them to reason and explain such a discovery especially in the proof process. Hölzl (2001) advocates this kind of task as a useful way to promote the natural realisation of the need for proof to verify geometric observation among students rather than explicitly instructing them to do so. He suggests an approach to task design where students are supposed to construct geometric situations in the DGS environment, such as a triangle with a circumcentre, orthocentre and centroid, leading them to discover the Euler line or to investigate the locus of the orthocentre of a triangle inscribed in a circle, inviting them to prove why such locus is a circle. This approach to task design adopts the surprising element of discovered geometric properties as the motivation for students to reason the cause of such behaviour, exploiting the DGS flexibility where the dynamic feature allows students to explore the geometric situation freely and effectively. Though Hölzl's initial intention with this task design is to promote the proof process, these tasks may in fact encourage students to adopt any type of possible reasoning strategy to explain the surprising property they observe.

2.7.5 Contradiction Investigation Task Leading to the Reasoning Process

Heuristic Exploration Task design discussed in the previous sub-section can be further complicated by adding an element of contradiction where the approach for one geometric case does not work with another. Hölzl (2001) proposes a more advanced approach to task design where students are supposed to construct a circumcircle of given quadrilaterals starting from a square or a rectangle to a cyclic and non-cyclic quadrilateral as well as irregular polygons. Students are supposed to realise that the strategy of drawing the circle by using the intersection of the two diagonals of the square and the rectangle as the circle's centre, does not work with some cyclic and non-cyclic quadrilaterals. The strategy also differs when constructing a circumcircle for an irregular polygon. Hadas, Hershkowitz and Schwarz (2000) also adopt the same approach to task design, allowing their students to encounter the fact that the relationship between the sum of interior and exterior angles and the number of sides of polygons actually differs. In these tasks, the contradiction between the students' intuitive anticipation of the geometric property and what they actually observe is used as a driver to orient students towards the reasoning process with geometric proof. Nevertheless, though the strategies of the participatant students in these studies happen to comply with the designers' intentions, there is no guarantee that other students would also follow these tracks and it is possible that they may not experience such intended contradictions at all if they choose to adopt a different strategy. Task design deliberately incorporating contradictory elements such as these, therefore, bases its rationale on an assumption that students will strictly follow such a path which may not necessarily be the case. The success of such a task is then partially dependent on the students' strategy of whether or not it complies with the designed task.

2.7.6 Uncertainty of Construction Task Leading to the Reasoning Process

Besides the contradictory cases discussed in the previous sub-section, uncertainty in a geometric construction problem can also be adopted as another task design strategy orienting students towards the reasoning process. In this kind of task, students are supposed to investigate the necessary conditions for a certain geometric construction and determine whether it is possible to construct such a figure. Hadas, Hershkowitz and Schwarz (2000) illustrate this type of task design with questions about the possibility of constructing a congruent triangle with given conditions, such as with one side and two angles equal to the given triangle, or to construct two congruent triangles with five equal elements (sides or angles). The students are also asked to explain why such a construction is possible or impossible, leading them to the reasoning process, especially proof. In this kind of task, students are supposed to refer to known geometric properties as the basis for their deductive reasoning to explain the existence or non-existence of the figure in question. The core of the task is, therefore, a geometric construction problem with additional conditions and the possibility of the construction as an extra tension, leading students to the proof process. Nevertheless, similar to the contradiction investigation task mentioned above, the aspect of 'uncertainty' in the task is very subjective, and it is possible that some students may not experience such 'uncertainty' at all during the task, especially when they are aware of such a possibility in the first place, or when they choose to adopt a different strategy from that anticipated by the designer. The key element of 'uncertainty', therefore, cannot be guaranteed in this kind of task, as it is very much dependent on the students' approach to the given problem.

2.7.7 'Black-box' Analysis Task

Another unique kind of task found in literature about DGS integration in the mathematics classroom is the 'black-box' task, where students are given completely constructed figures and they are supposed to analyse them to find out what kind of figures they are. One experienced participant teacher in Laborde's study (2001) mentioned in Sub-section 2.7.1 designed a task where students are supposed to study and analyse a pair of figures to determine what kind of transformation is used. The 'black-box' task is a useful assessment tool to evaluate students' understanding of the concepts taught. It asks students to apply their geometric knowledge from past lessons to a new geometric situation in order to figure out how the given diagrams are constructed. Besides geometric knowledge, their familiarity with the software features also plays an important role in helping them to figure out the construction process.

This section surveys a range of DGS task design approaches adopted by teachers and scholars. It illustrates the flexibility of DGS as a pedagogic tool that can be integrated into geometry education in a number of ways. These types of DGS task designs for students provide a basis for the task development phase of this research to be further discussed in Chapter 7.

2.8 STUDENTS' REACTIONS AND INTERPRETATIONS OF DGS FEATURES

With its range of unique characteristics, working in the DGS environment can provide a totally new experience which users need to familiarise themselves with in order to understand and control the software. This leads to the interpretation of DGS features which do not necessarily conform to the software designer's intentions. Certain research on DGS use focuses on students' reactions and interpretations of DGS features, especially towards drag-mode, parent-and-child relationship and dynamic measurement. They will be discussed in turn based on each of these features.

2.8.1 Dynamic Property of Drag-Mode

Notable research focusing on the way the student uses DGS' drag-mode in geometric task is conducted by Arzarello et al. (1998; 2002). In this research, Arzarello et al. examine how the drag-mode plays a role in the student's exploration of the internal quadrilateral constructed from four intersections of angular bisectors of another quadrilateral. Students are supposed to use DGS to explore conjecture and try to empirically validate the conjecture. Students' dragging behaviour is analysed and can be identified into two distinct processes:

"1) ascending processes, from drawing to theory, in order to explore freely a situation, looking for regularities, invariants, etc.

2) descending processes, from theory to drawings, in order to validate or refute conjectures, to check properties, etc."

(Arzarello, et al., 2002, p. 67)

During the task, it was observed that students kept switching between these two processes of ascending and descending from figures to concepts or from the empirical to the theoretical level. This shift is identified by Arzarello et al. as a process of 'abduction' or generating a hypothesis which is not yet a statement and needs further supporting or negating empirical evidence to verify or reject. It can be considered 'creative abduction' as defined by Eco (1983), where a rule is invented based on figure observation in a dynamic environment.

Besides identifying these processes, Arzarello et al. also classify a hierarchy of the dragging modalities into different kinds of dragging, e.g. 'Wandering dragging': moving the basic points on screen randomly; 'Bound dragging': moving a semi-draggable point along the path of an object, 'Guided dragging': moving the basic points of a drawing in order to give it a particular shape; 'Dummy locus dragging': moving a basic point so that the drawing keeps a discovered property and 'Line dragging': moving points along a line in order to keep the regularity of the figure.

This study shows that the drag-mode in DGS can provide a wide range of strategies the learner can use in the geometric exploration process and Arzarello et al.'s classification of dragging behaviours provides a helpful model to understand students' thinking during each stage of task. These rich modalities of drag-mode in DGS support the notion that this software provides a highly flexible platform for geometric task.

2.8.2 Dependency of Parent-and-Child Relationship

Some early research on DGS placed the focus on examining the effect of parent-and-child relationships in DGS on users. Studies by Healy, Hölzl et al. (1994) and Hölzl, Healy et al. (1994) observe how students with limited or no experience with DGS perform construction tasks where they are supposed to use the parent-and-child relationship to construct a figure which cannot be “messed-up” by dragging. This is a useful skill in figure construction in the DGS environment, especially when students are supposed to use such a figure to explore geometric relationships. Both studies show that some students at around age 8 are capable of understanding the concept of the parent-and-child relationship since they could successfully draw pictures or construct geometric shapes that will retain the desired properties when dragged.

Jones (1996, 1998) also examines how a pair of 12-year-old students who have prior experience with the DGS environment define the parent-and-child relationship between objects in Cabri. When given tasks to construct geometric figures such as different types of quadrilaterals or a diagram of intersecting circles on paper, students needed to appreciate the parent-and-child relationship in DGS in order to construct diagrams which cannot be ‘messed-up’. They first realise such a relationship when some objects also disappeared once a certain object was deleted and they called this relationship ‘dependency’ following the word given by the researcher. Key findings from this study show that students developed their own interpretation of the phenomena encountered in the DGS environment and they usually created their own names for such interpretation. For example, they used the phrase ‘all stay together’ to refer to the desired properties of a diagram which cannot be messed-up, and referred to the radius point of a circle an ‘edge point'. Interestingly, they falsely interpreted the fact that an intersection in Cabri is immovable as a result of an intersection point, 'gluing' elements of the whole element together. In fact, the intersection point is immovable because of the one-way dependency. This interpretation shows that students tend to relate the behaviour in the DGS environment to their experience in everyday life, which may be contrary to the real underlying mechanism of the software.

Talmon and Yerushalmy (2004) go further in examining the user's interaction with this parent-and-child relationship. They interviewed a pair of 9-grade students and a pair of mathematics education graduates to elicit their reaction to the dynamic behaviour of the figure when a certain point is dragged. The result shows that they often guess the dynamic behaviour in DGS incorrectly by expecting a reverse order, where moving a child affects the parents. It is analysed that this notion stemmed from the user's reference to a paper-and-pencil environment where the order of figures does not exist. This finding shows that the behaviour of the parent-and-child relationship in dynamic geometry software does not necessarily conform to the user's intuition, even for the experienced.

Reports of this research demonstrate that students are capable of understanding the parent-and-child relationship at the level needed for constructing a robust figure which cannot be "messed-up". However, the dynamic behaviour of the figures, resulting from the parent-and-child relationship can be unintuitive for users when they are supposed to predict the changes. As yet, there appears no research which directly focuses on the users' interpretation of the parent-and-child relationship in order to provide a clearer understanding of how the learner perceives this mechanism in the DGS environment.

Though the parent-and-child relationship in DGS is not part of the Euclidean geometry it portrays, its inevitable presence in DGS makes it an important factor which students need to understand in order to distinguish it from the intended geometric concept to be learned in the task. However, Mariotti (2000) suggests that the student's experience with the parent-and-child relationship in the DGS environment can actually provide a metaphor for the concept of logical dependency; a fundamental reasoning process in deductive proof in geometry. This claim is interesting to pursue in order to find out whether and how the software working process can influence the student's development of reasoning skill.

2.8.3 Utilising Dynamic Measurement

With the intention of extending the study by Arzarello et al. on the student's reaction to DGS drag-mode (1998; 2002), Olivero and Robutti (2007) conduct similar research which places the focus on dynamic measurement in DGS instead of drag-mode. Their research was set to examine the15-16 year-old students' approach to measuring, and the cognitive role of measurement in geometric proof activities in the DGS environment. The analysis of the data gained from classroom observation in a number of schools shows that students adopt a number of different modalities similar to the case of drag-mode in Arzarello et al.'s study. They divide measuring modalities in the DGS environment into two main categories. The first category includes modalities which relate to the shift from the spatio-graphical field to the theoretical field, similar to the ascending process in Arzarello et al.'s study. These include 'wandering measuring', i.e. using measurement randomly without a precise plan, 'guided measuring', i.e. using measurement deliberately to examine particular cases one after the other and 'perceptual measuring', i.e. using measurement as a means of checking the validity of intuition-based perception. The second category conversely includes modalities which relate to the shift from the theoretical field to the spatio-graphical field, similar to the 'decending' process in the Arzarello et al. study (ibid.). They include 'validation measuring', i.e. using measurement to check the formulated conjecture in order to either accept or refute it, and 'proof measuring', i.e. using measurement to gain further support after the process of proof. The measure feature in DGS can, therefore, be a helpful tool for the students' process of justification, especially by abductive reasoning, similar to the case of the drag-mode feature.

2.9 DGS AND LEARNER'S GEOMETRICAL REASONING

This section discusses literature about the effect of DGS features on the student's geometric reasoning process. It will focus on research examining the influence of the dynamic feature on the student's strategies of reasoning.

The dynamic feature, together with the parent-and-child relationship in DGS allows the user to generate countless examples of certain geometric cases in order to observe the variant and invariant properties. This exploratory power of DGS inspires many researchers to examine how the dynamic feature in DGS can be used to promote and support students' reasoning in the geometric proof process.

Jones (2000) examines the foundation stage of reasoning which can lead to formal proof where 12 year-old students were supposed to construct a robust quadrilateral with DGS and classify it by inclusive properties portrayed by the software. Although there is evidence that the students made progress from everyday language description to mathematically precise explanation, the study also shows that some students faced problems in quadrilateral classification tasks. This stems from the fact that students are fixated on exclusive definitions of geometric shapes, e.g. a square is not a rhombus while the dynamic feature in DGS advocates inclusive definition, e.g. a rhombus can be dragged to make it look like a square. The research result, therefore, is affected by the incompatibility of the student's pre-requisite knowledge and it was not made clear how DGS actually plays a role in the students' reasoning of quadrilateral classification in this case.

Marrades and Gutiérrez (2000) also investigate students' progress from empirical to abstract justification in a series of geometric problems requiring the deductive reasoning process. The result indicates that only students in the higher-attainment group show such progress and the process is much slower in the group of less-able students. This research also points out that one of the most important factors for a successful geometric proofing process is the student's possession of the requisite prior knowledge for such a task. A notebook containing all that had been learned was shown to be as valuable as DGS tool itself.

In the same series of research on the role of DGS in the proofing process, Hadas, Hershkowitz et al. (2000) conduct another study focusing on the role of contradiction and uncertainty in promoting the need for proof in the DGS environment, with the properties of sums of internal and external angles of polygons task, designed to surprise students with unexpected results. Though the strategy used in this research study was shown to successfully attract students and encourage them to perform a formal proof, it is still arguable that the success of a lesson depends on the way it is designed rather than on DGS features. From these attempts to demonstrate how DGS can be used to promote proof in various geometric activities, there is a general notion from all these research conclusions that the role of DGS remains a tool for exploration without a direct affect to the desired process.

Drawing from these studies, Laborde (2000) concludes that though DGS provides a flexible platform where students can explore the geometric situation in the way they wish, the more important factor that can lead to a successful proof process is the way the teacher designs the 'milieu' or environment of such activity. Later research by Connor, Moss et al. (2007) with secondary mathematics pre-service teachers who already understand the role of formal proof also supports this notion, when those pre-service teachers actually chose to use DGS less and less when they try to justify mathematical statements deductively with proof.

Another interesting study conducted by Chazan (1993) investigates the student's rationale for problematic belief that empirical evidence is proof and deductive proof is simply evidence. Chazan interviews a number of high school students after lessons on geometric statement verifications. The verification includes the empirical process of dynamic measurement in a DGS called Geometric Supposers and corresponding deductive proof. He finds that a certain number of students fell into the trap of general misconception, reversing the role of empirical measurement as evidence and deductive proof with confirmation. Students who believe that empirical measurement is a confirmation think that if one could collect data for a certain range of types of geometric figures; for example, if a case about a triangle can be explored for acute, obtuse, right, equilateral and isosceles triangles, it should be sufficient to generalise to all other triangles. This stems from an idea that a single case of each type of triangle should represent the result for all triangles in the same category, reflecting the strategy identified by Balacheff (1988) as ‘Generic Example’, discriminating each type of triangle.

On the other hand, students who believe that deductive proof is just evidence which cannot be generalised for other cases, view the process of deductive proof as a verification of just a single diagram. They also believe that deductive proof would not provide safety from counterexamples. One student in this group even raised a language issue stating that deductive proof should not be able to represent more than a single case since the language used in the proof process is always in singular form. These findings provide a very useful insight into the issue of the student’s individual understanding of ‘proof’ as already suggested by Balacheff (2008). In order to examine the student’s deductive reasoning skill, the ability to give a formal proof to a geometric situation should not be used as the sole criterion, since the concept of proof can be interpreted in a number of different ways. The syllogistic relationship between premises should, therefore, provide a better reference to the students’ ability in deductive reasoning, though some of them may not be able to draw this skill to the activity of geometric formal proof successfully.

A recent research by Leung (2009) looks more closely at the role of the dynamic feature in the DGS environment on students’ construction and proof tasks of a particular geometric diagram. From a task-based interview transcript with a 16 year-old Hong Kong student working on a diagram of a square inscribed in a regular polygon, which he is supposed to construct and provide proof for its validity, Leung demonstrates that the student’s written proof clearly shows a reference to dynamic property found in the DGS environment where a prime sign is used to signify points which can be dragged. The student also uses an equation form to suggest that the statement is true, no matter how such point (with a prime) is dragged. Leung’s research clearly shows the genuine case where DGS plays a role in the student's geometric proof process when variant and invariant properties of the diagram become an integral part of validation. Even though deductive proof was the key focus in this study, an analysis of the transcript presented in the paper also shows evidence of different modes of reasoning adopted by the students. He uses a snapshot of the case to explain the geometric relationship among elements with the intention of generalising for other situations which can be considered a case of 'Generic Example'. He also formulates a conjecture when he visually observes that a locus of a particular point is a straight line which can be deemed to be a process of abductive reasoning. Moreover, at one point, he even asserts a claim by intuition rather than valid evidence, indicating the adoption of illogical reasoning, despite his talent in the subject. This study therefore shows a range of reasoning strategies which a student may use in geometric task and some of these can have a direct relationship with the features in the DGS environment.

This chapter surveys important topics in relation to the use of digital technology, especially the dynamic geometry software to cultivate the students' higher-order thinking skill of reasoning. It discusses the concepts in the hierarchy of thinking skills, definitions and interpretation of three key types of reasoning; the concept of argumentation; how reasoning plays a role in geometric education; and a section about the features of dynamic geometry software and relevant research. It can be seen that most of the research about adopting DGS in geometry education concentrates on how it can be used to promote deductive proof with little focus on other types of reasoning. This leaves a knowledge gap in how DGS can play a role in other types of geometric reasoning, especially in how its unique features can be used for geometric justification. This research will, therefore, concentrate on this under-studied area, in order to examine how the DGS environment can support students' reasoning process and how it may help cultivate students' higher-order thinking skill.

3 THEORETICAL FRAMEWORK

The focus of this research is on the relationship between the DGS environment and students' higher-order thinking. The central elements involved in this study are; the use of digital technology as an educational tool, and the students' higher mental function. As discussed in the previous chapter, different scholars may have different views on human higher-order thinking. However, the notion of higher-order thinking, which has an explicit relationship with the concept of the mediating tool, is found in the works of Vygotsky. His approach should, therefore, reflect the relationship between the development of higher-order thinking, and using digital technology as the available tool. This chapter examines my proposed use of the Vygotsky's concept of tool use, and its connection with higher mental operation. It will also look at a specific activity model, based on Vygotsky's idea, before a model of study developed for this particular research is presented.

3.1 VYGOTSKY'S RELATIONSHIP OF TOOL USE AND HIGHER MENTAL FUNCTION

For Vygotsky, tool use emerges when a subject organism changes from an unsuccessful direct action towards an object, to application of a device as a tool in order to achieve the same goal (Vygotsky, 1930). The diagram of Vygotsky's model of tool use is shown in Figure 3.1.

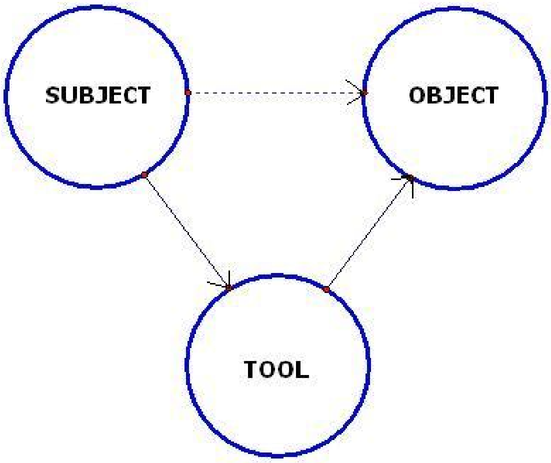


Figure 3.1 Tool use as mediated activity

From this model, the direct connection between subject and object is replaced by two connections; between the subject and the tool, and between the tool and the object. The subject, therefore, needs to control the tool in order to interact with the object. Vygotsky calls this inclusion of a device 'mediated activity' and claims that this indirect activity totally changes the interaction between the subject and the object. It modifies the whole working process and abolishes the need for natural, direct action (shown as a dotted connection in Figure 3.1). The instance of tool use, therefore, provides a different scenario from the case of natural direct action, although they both lead to the same goal.

Apart from physical tools, Vygotsky also introduces others, namely 'psychological tools'. While physical tools involve the application of a tangible object as a device to achieve the desired goal, psychological tools, on the other hand, are artificial formations directed towards the mental process. Examples of psychological tools include; the use of signs, language, symbol, counting, numeration, scheme, map, diagram etc. (ibid.). Among these psychological tools, Vygotsky considers signs and language to be the most eminent factors in human higher mental development. As with physical tools, the use of signs and language or other psychological tools is also ‘mediated activity’, with psychological tools replacing the direct connection between stimulus and response (Vygotsky, 1978). The structure of sign and language operations as psychological tools is depicted in Figure 3.2.

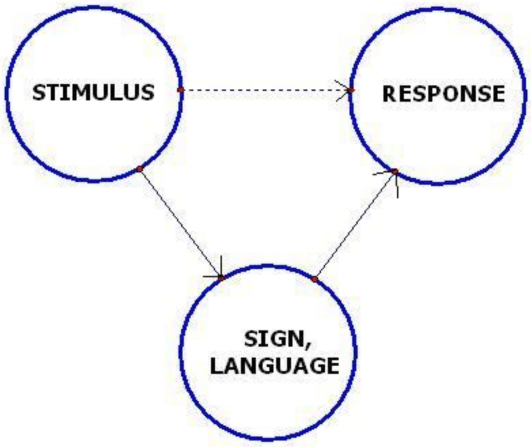


Figure 3.2 Sign and language operation as mediated activity

As in the case of physical tools, the presence of sign and language operation modifies the structure of mental operation from a direct stimulus response action to a more complex mediated act. In this process, the direct impulse to react is therefore inhibited, and replaced by the indirect operation of sign and language use. This type of organisation differentiates between human and animal behaviour and is a basis for higher psychological processes (Vygotsky, 1978).

Though both physical and psychological tool operations are categorised as mediated activity, the way they orient human behaviour are actually converse. While a physical tool serves as a conductor of human influence to change the object of activity, a psychological tool, on the other hand, does not change the object of a psychological operation. It acts as a means of internal activity aimed at mastering oneself. A physical tool is, therefore, externally oriented, while a psychological tool is internally oriented (ibid.).

The link between Vygotsky's concept of a mediating tool and higher mental function can be found in his assertion of language as a psychological tool which mediates human's higher mental operation. Vygotsky claims that speech plays an essential role in the organisation of higher psychological function. This speech is a result of an internalisation of social speech where its function is changed from communication to mediation for our thinking. Vygotsky called this internal speech 'verbal thought' (Vygotsky, 1978) and his theory indicates that this verbal thought is actually the medium that conveys a person's thinking.

In terms of using digital technology as a physical tool for learning geometry, Vygotsky's model of tool use suggests that the inclusion of digital technology, DGS in this case, will change the way learners interact with geometrical knowledge through the tool's mediation. The geometry knowledge is, therefore, mediated by the DGS tool and the learners need to interact with such knowledge via the commands in DGS. This scenario suggests two levels of learning; the DGS tool and geometrical knowledge interaction. The first level concerns the concrete part of activity where learners physically deal with the software in order to control the situation through a range of commands. The second level involves the abstract part of activity where learners mentally deal with the geometric properties they encounter from the dynamic diagram with mediation by verbal thought. Vygotsky's model, therefore, highlights the process of mediation, which makes learning in the DGS environment a unique experience.

3.2 INSTRUMENTED ACTIVITY SITUATION MODEL

Vygotsky's idea of tool use provides a foundation for several theoretical activity models developed by later scholars for different specific purposes. One model which focuses on the user's psychological operation during the act of tool use is proposed by Verillon and Rabardel (1995) in their Instrumented Activity Situation (IAS). Unlike other models such as Activity Theory (Jochems, van Merrienboer, & Koper, 2004), where Vygotsky's model of object-subject-tool is extended to reflect organisational and pedagogic perspectives through inclusions of rule, community and division of labour, Verillon and Rabardel's IAS model places more emphasis on the user's utilisation scheme by taking a close look at how an individual interacts with the tool in order to achieve the desired goal. This idea better conforms to the aim of this research, where the relationship between the technological environment and the learner's cognitive activity is the main focus, and not its connection to the surroundings. In order to examine this utilisation scheme, Verillon and Rabardel identify the process of physical tool use into two separate concepts; an 'artefact' or man-made object, whether material or not; and 'instrument', which is a user's psychological construct of the artefact. The tool, whether physical or psychological, cannot become an 'instrument' by itself without the mental action from the user. The relationship between, subject, object and instrument is shown in Figure 3.3.

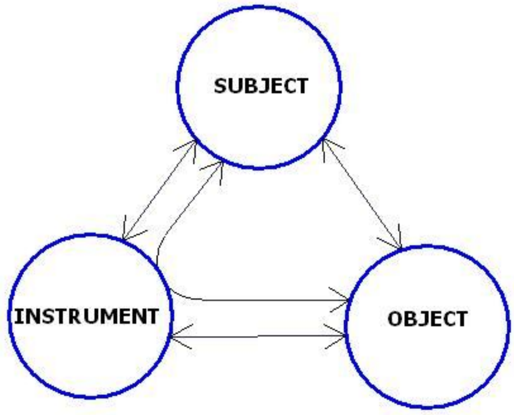


Figure 3.3 Instrumented Activity Situation (IAS) model

It is notable that all the connections between subject, object and instrument are now bidirectional even in the case of physical tool use. This illustrates that the relationships among these three entities are interactional, highlighting the fact that each of these entities can influence the whole scheme of operation. For example, the properties or characteristics of object and instrument can affect the way a subject controls the tool. Another addition to the IAS model is the special arrow linking the subject and the object via the instrument, indicating an indirect relationship between them. This relationship is mediated by the instrument and does not necessarily match the direct interaction presented by the direct two-way arrow between them.

Verillon and Rabardel (1995) exemplify this mediated relationship in a case where a child learns to use a spoon to carry milk and to carry mashed potatoes, where the behaviour of liquid and rigid food interact with the child in relation to the way a spoon is used to carry them. This special connection highlights the concept of the ‘utilisation scheme’: the framework of action that the subject organises in order to use a certain instrument. The utilisation scheme is an individual behaviour and can vary from person to person and hence does not necessarily need to conform to the original purpose of the tool's design. It depends more on how each user constructs such an instrument from the artefact (ibid.). The utilisation scheme in the case of using DGS to learn geometry can reflect the learner's higher mental process during the activities. The way students utilise DGS features to collect empirical data, and the way they utilise such data to verify a geometric statement can outline their reasoning process. On the other hand, their limited utilisation scheme of the DGS tool to gain the desired data in order to validate conjecture, can also disrupt their reasoning process. The analysis of the student's utilisation scheme of the DGS tool, especially in how it relates to the characteristic of the tool and the objective of the task is, therefore, essential for the focus of this study.

In order for the IAS model to be applicable to Vygotsky's concept of the psychological tool, as well as the physical tool, Verillon (2000) later elaborates the IAS model to illustrate the instance of psychological tool operation. However, the fact that Vygotsky refers to psychological tools as semiotic instruments, i.e. instruments which carry meanings (language, symbol, counting, numeration, scheme, map, diagrams) brings about an issue concerning the autonomy of a sign and its meaning. Saussure (1983) distinguishes the components of a sign into 'signifier' and 'signified'. 'Signifier' means an object which is used to refer to something either in the form of words, gestures or visible shapes, while 'signified' is the mental concept. Saussure asserted that 'signified' and 'signifier' are not necessarily connected and are two independent entities. The connection is later established by an individual when the meaning of each sign is learned. This idea explains the case where a single 'signifier' may carry more than one 'signified' component such as words with different meanings, and the case where a different person may connect the same signifier to a different mental concept. To take this distinction between signified and signifier into account, Verillon proposes that the meaning part of the instrument in the IAS model should be isolated from the instrument itself. He termed this meaning part of the instrument ‘referent’ and added it as the fourth entity of the model. Verillon’s identification of the ‘referent’ element in the case of psychological tool use suggests that the instrument and its referent may have different influences on subject and object entities, and therefore should be distinguished. This ‘referent’ element also has an interactional relationship with all other entities, including the instrument that portrays it. The modified IAS model for the case of psychological tool operation is shown in Figure 3.4.

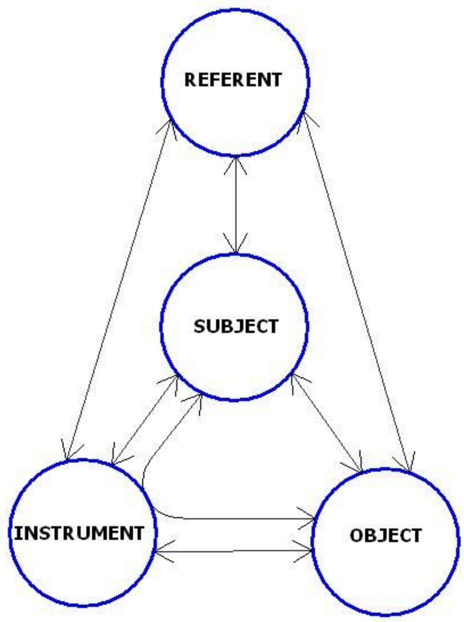


Figure 3.4 Modified IAS model for psychological tool operation

This modified model clearly shows that the subject can react to the instrument and its referent differently, and the way the instrument portrays the referent can also influence the whole process of operation. Moreover, the referent can also have a direct effect on the objective of the task, highlighting the independence of meaning from its carrier.

Verillon's modified IAS model for psychological tool operation depicts very detailed interrelationships between subject, object, instrument and the referent it portrays. It emphasises that any of these entities can play a significant role in the overall process. It also identifies the indirect interaction between entities which do not necessarily conform to the direct interaction, especially from the viewpoint of the subject. This modified IAS model should, therefore, provide a useful framework for analysis of the student's use of DGS as a psychological tool to learn geometry.

3.3 MODEL OF STUDY

Based on the model of psychological tool use proposed by Verillon (2000) in the previous section, the model to be used in this research has to be established. For the case of the learner's use of DGS to learn the concept of geometry, the subject would be the learner, who will have an opportunity to use the tool themselves, the object would be the objective of the learning activities they are to perform, and the instrument would be the DGS tool. Since the prime purpose of the DGS tool is to provide a platform of Euclidean geometry in the dynamic mode, it should be reasonable to interpret DGS as a psychological tool that portrays the concept in Euclidean geometry. The referent entity in this case should be the concept of Euclidean geometry. Note that DGS as an instrument entity in this case would account for all of its concrete parts including the diagram displayed on screen, while the Euclidean geometry as a referent entity will account for the abstract part of the concept. The model of study for this research is shown in Figure 3.5 with the central Learner entity placed in the middle of the model interacting with other entities.

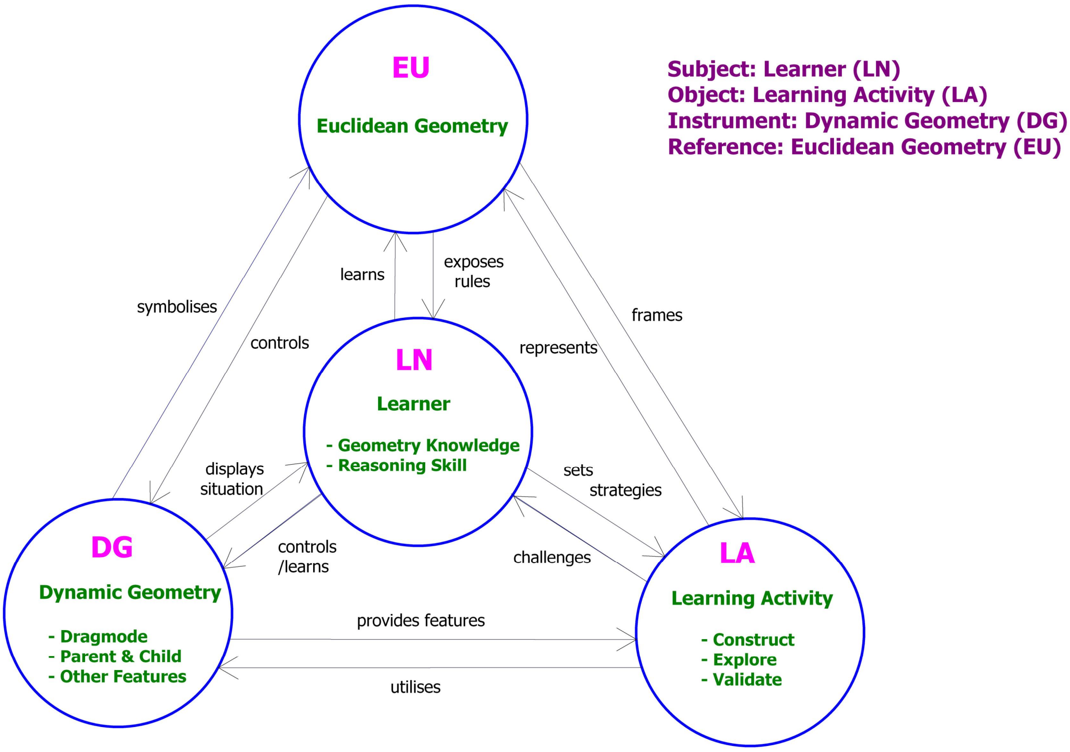


Figure 3.5 Model of study

Note that the indirect relationship between Learner and Learning Activities through the DGS tool is not yet present in this model. The indirect relationship between all these elements is discussed again in Chapter 6. The detail of direct relationships between each entity is discussed as follows:

LEARNER (LN) - LEARNING ACTIVITY (LA)

The learning activity will be used to stimulate the learner's learning process. The main objective of the learning activity is set to investigate the learner's reasoning strategies. The main learning activity would be in the form of a geometric construction and exploration task where the learner will be challenged to verify their construction, as well as the validity of geometric conjecture observed from the exploration.

LEARNING ACTIVITY (LA) - EUCLIDEAN GEOMETRY (EU)

Though the objective of the learning activity in this research may aim to investigate the learner's reasoning process, the objective of the learning activity itself would aim at the learner's acquisition of the Euclidean geometry concept, especially its deductive-axiomatic nature, through the process of formal proof. The learning activity would, therefore, be designed to cover this characteristic of Euclidean geometry, though the student's options of strategy will be left open.

LEARNER (LN) - EUCLIDEAN GEOMETRY (EU)

The relationship between learners and Euclidean geometry is rather unique in this model, in the sense that it may or may not be established, depending on the learner's performance in the learning activity. Though the objective of the learning activity aims at the learner's acquisition of knowledge from Euclidean geometry, this part of connection is an expectation rather than what will happen in the learning situation. The concept of Euclidean geometry will display itself through the rules governed in the learning activity. It is up to the learners whether or not they can grasp this concept and relate it to their prior knowledge in this domain. The learner's background in Euclidean geometry is, therefore, another important factor which needs to be examined before commencing the task, as it can affect what they can or cannot learn from the activities.

LEARNING ACTIVITY (LA) – DYNAMIC GEOMETRY (DG)

The central role of the DGS tool in this model of study is the 'mediator' between learners and learning activities. The learner's interaction with the learning activities will be mediated by this tool. The learning activity will adopt features offered by the DGS tool to provide the learner's flexibility to choose their own strategies to approach the task. Unique features, such as drag-mode, can support the breadth and variety of the activities in order to allow a range of strategies by the learners. The capacity of DGS, i.e. what it can and cannot do will determine the range of learning activities that can be conducted with the tool.

DYNAMIC GEOMETRY (DG) - EUCLIDEAN GEOMETRY (EU)

In this model of study, the DGS tool is considered an instrument for portraying Euclidean geometry as its referent. The relationship between the DGS tool and Euclidean geometry would be a carrier of meaning, and the meaning itself. It should be noted that Euclidean geometry can be portrayed in a different medium or environment, such as by diagrams in the paper-and-pencil environment, or by the verbal process of formal proof. The way DGS portrays Euclidean geometry can give a unique representation of Euclidean geometry different from any other medium or environment. The distinct feature of DGS in this respect is its dynamic feature; this feature makes diagrams in DGS qualitatively different from diagrams in the paper-and-pencil environment in such a way that it can be temporally manipulated by the user. This flexibility provides a new path in Euclidean geometry which can be experienced by the learner. DGS, therefore, does not only present 'Euclidean Geometry' it also presents a unique way of 'Euclidean Geometry' portrayal.

LEARNER (LN) – DYNAMIC GEOMETRY (DG)

The interaction between the learner and the DGS tool is a key source of information for this research. It can be seen from this model that by dealing with DGS in this geometric activity, learners will encounter three distinct components simultaneously:

1) The learning activity as mediated by DGS.

2) Euclidean geometry as portrayed by DGS.

3) The internal mechanism of DGS itself, such as command algorithm, the parent-and-child relationship and its dynamic feature.

Learners' understanding of the tool's behaviour and utilisation scheme of the tool plays an important role in this part of the relationship since it can strongly affect the whole learning process and reasoning strategies. The Learner's perception of these unique DGS features, and how this perception affects their reasoning strategy, forms the basis of the data for this research.

With the identification of these four independent entities, i.e. learner, learning activity, dynamic geometry and Euclidean geometry in this proposed model of study, it should be much more convenient to analyse the interrelationships among these key entities. Putting the 'learner' as a subject should help to see how their reasoning strategies relate to all other entities, especially when the learner is working in the DGS environment. Moreover, how the other entities interrelate and how such interrelationships influence the learner's reasoning, can also be distinguished in this model.

This proposed model should therefore be helpful in eliciting the learner's reasoning processes in this technological environment.

However, this model depicts only direct relationships between these four key entities. The 'mediated' relationship between each entity, especially between the learner and the learning activities through DGS, as suggested in the original IAS model is not yet presented. All the mediated relationships will be refined and discussed again in Chapter 6, where the model for data analysis is developed. The model of study given in Figure 3.5 therefore provides the basic outline of key entities and their direct relationships, and it will be set as a theoretical framework for this research.

4 RESEARCH QUESTIONS

Since the model of the research framework has been established in the previous chapter, the research questions can now be formulated to identify key aspects that this research proposes to examine. The main research question of this study is:

“What is the relationship between the learner’s higher-order thinking skill of reasoning, the DGS environment, and DGS-based geometric tasks when challenged by a reluctant believer, and how these relate to the learner’s acquisition of Euclidean geometry knowledge?”

From this question:

The term ‘DGS environment’ means a computer environment generated as a screen interface with a control system of graphical software to simulate the dynamic model of Euclidean geometry (Hölzl, et al., 1994; Sträßer, 2002). For software to be called ‘dynamic geometry software’ it should provide these three basic elements:

(1) The construction module for the user to construct geometric objects.

(2) The function where the property of the new object can be defined in relation to the existing object(s).

(3) The drag-mode where the user can move any part of the figure, while geometric and pre-defined relationships retain.

‘Reasoning strategies’ covers all activities providing information to support one’s beliefs or claims. It can be illogical (e.g. ‘because I think it is’) or logical, and can be inductive, deductive or abductive.

Based on the model of the study presented at Figure 3.5 in the previous chapter, the research question can be broken down into three sub-questions according to the interaction between the learner and other three entities; dynamic geometry, Euclidean geometry and learning task. This should help me as a researcher focus on each relationship more closely in order to answer the main research question. These three sub-questions are formulated in the following three sections.

4.1 RESEARCH SUB-QUESTION 1

“How do learners use their geometry knowledge to reason about the way DGS portrays Euclidean geometry?”

This sub-question concentrates on the inter-relationship between Learner-Dynamic Geometry-Euclidean Geometry. It aims to examine the learner’s perception and interpretation of how DGS portrays Euclidean geometry, compared to their prior knowledge of Euclidean geometry in the paper-and-pencil environment. This question focuses on how learners use their reasoning in order to make sense of the way DGS presents Euclidean geometry. Learners will be involved in construction and exploration tasks. They will be asked to observe how DGS commands function in constructing and exploring geometric figures, and explain the rationale. Two distinctive features in DGS will be highlighted; drag-mode and the parent-and-child relationship. Besides introducing learners to drag-mode in construction and exploration tasks, the concept of the dynamic figure which cannot be 'messed-up' will also be asserted during the session. This will provide an opportunity for learners to discover the parent-and-child relationship, or dependency of movement between elements, in order to see their reaction and reasoning strategies to explain this behaviour in the DGS environment.

4.2 RESEARCH SUB-QUESTION 2

"What kind of reasoning strategies do learners adopt in geometric construction and exploration tasks in the DGS environment?"

This sub-question concentrates on the inter-relationship between Learner-Learning Activities-Dynamic Geometry. It aims to examine the learner's reasoning strategies towards construction and exploration tasks in the DGS environment. In the construction task, learners should verify that the constructed figure conforms to the given instructions. In the geometric exploration task, they should use the drag-mode to explore pre-constructed figures, conjecture and then verify a geometric statement with reason. The Learner's reasoning will be identified as to whether it is inductive, deductive, abductive or a combination of these. Reasoning will be categorised as 'inductive' if the learner uses only empirical data to support their claim. For 'deductive' reasoning, the learner should support the validity of the statement by referring to known geometric property to some extent. They are not expected to produce 'formal proof' of the whole situation as already discussed in Sub-section 2.4.1, in that the concept of 'proof' in geometry education is still ambiguous and arguable. The focus would, therefore, be placed on the process of 'deductive reasoning', rather than the whole scheme of 'proofing'. Moreover, emphasis will be given to validity, rather than truthfulness. Even if the geometric property the learner refers to is incorrect, it would still be considered as deductive reasoning, as long as reference has been made to a certain property. In the case of 'abductive' reasoning, the learner is supposed to form a hypothesis to explain the situation, and provide further evidence to support this hypothesis.

This part of the question examines what, and how, features in DGS are used to support the learner's reasoning strategies. For example, the variant and invariant behaviour under drag-mode may be used as empirical evidence for inductive reasoning; or the discovery of a certain pattern of a geometric situation in the dynamic environment may lead to the learner's formulation of hypothesis in abductive reasoning.

In order to investigate the learner's ability to reason in each circumstance, the view that reasoning should be able to convince others is asserted in pursuing this research question. If their reasoning is not strong, the learner will be challenged by the researcher to provide better reasoning to support their claim. The challenge is expected to influence the learner's reasoning strategy, in order to see how they may utilise features in DGS to provide a better explanation.

4.3 RESEARCH SUB-QUESTION 3

"What approach to task design will enable the learner to use reasoning strategies to acquire knowledge in Euclidean geometry?"

This sub-question concentrates on the inter-relationships within Learner-Learning Activity-Euclidean Geometry and involves the design process of learning tasks used in this research. This sub-question seeks criteria to determine the appropriate task design approach to encourage the learner's reasoning strategies in the geometric verification process. Since the sheer flexibility of DGS allows it to be integrated into geometric learning activities in a variety of ways, task design approaches that challenge and encourage reasoning strategies from the students should be carefully considered. The process of task design includes the selection of suitable topics from Euclidean geometry, as yet unknown to the students, and establishment of tasks which can encourage reasoning strategies to verify such rules. The possible utilisation of features in DGS in such task should also be taken into consideration during the design process. Prior knowledge needed for certain reasoning strategies should also be outlined. This leads to an important issue in geometric reasoning activities, such as, whether the relevant knowledge or properties should be given to learners to facilitate their reasoning strategies, and determine what should be provided for students in performing these tasks. The answer to this sub-question should help shape the research instruments in order to best stimulate the learning situations deemed desirable to the investigation of the main research question.

4.4 RESEARCH SUB-QUESTIONS IN THE MODEL OF STUDY

These three research sub-questions can be mapped back to the triangular relationships between the four entities with the learner as the central entity. Research sub-question 1 involves the relationships between LN-EU-DG. Research sub-question 2 involves the relationships between LN-DG-LA while research sub-question 3 involves the relationships between LN-LA-EU. The positions of the three research sub-questions: research sub-question 1 (RQ1), research sub-question 2 (RQ2) and research sub-question 3 (RQ3) are shown in Figure 4.1.

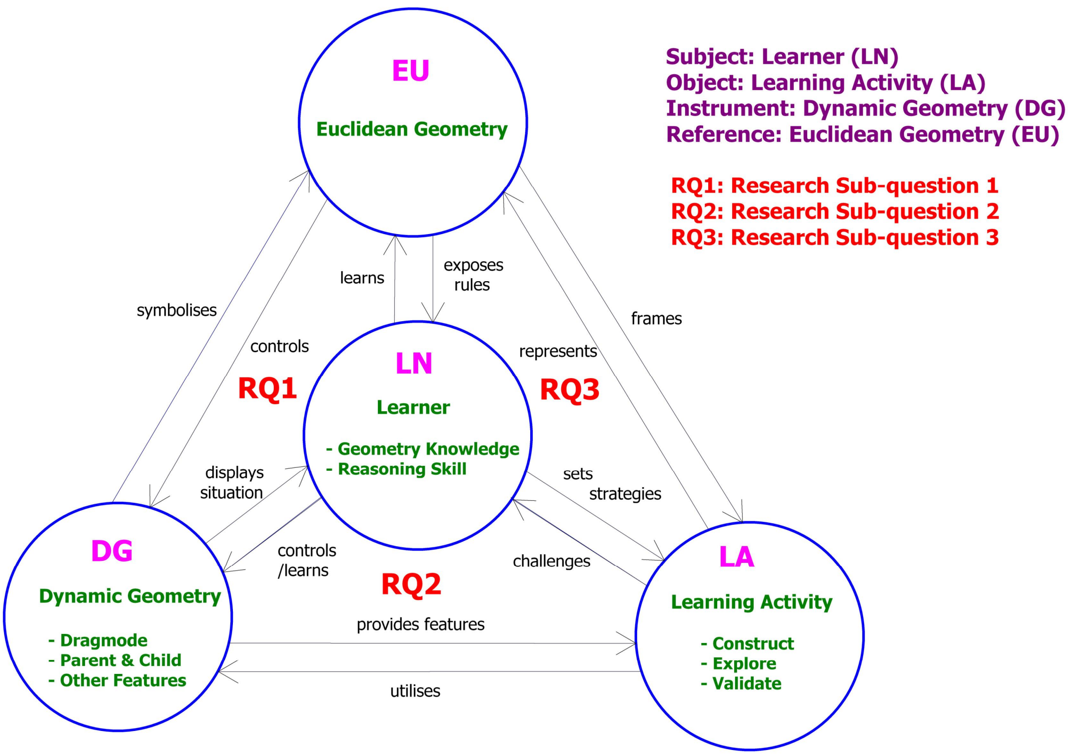


Figure 4.1 Model of study and research sub-questions

These three sub-questions would be set as fundamental queries of this research where research methods, data collection and analysis strategies would be designed in order to seek answers. The answers to these three sub-questions would then be accumulated in order to answer the main research question, bringing out the relationship between the learner, DGS and Euclidean geometry through the learning tasks designed by the researcher.

5 RESEARCH DESIGN

This chapter outlines the overall design of this research. It discusses the rationale for the choice of interpretivism as an epistemological stance and the case study as the research methodology. It also discusses the data collection techniques adopted in the research. The criteria for choosing the research setting, the choice of school and students are also explained. The chapter concludes with the types of data obtained from the empirical phase and the basic strategy employed to analyse that data together with validity and trustworthiness of the research.

5.1 INTERPRETIVISM AS THE RESEARCH EPISTEMOLOGY

Based on the discussion in the LITERATURE REVIEW chapter, it can be seen that, even on the same topic, different scholars may have alternative interpretations of the same term or concept, depending on their experiences, personal opinions or viewpoints. This diversity shows how knowledge, especially in social sciences, can be complex and multifaceted where an individual's background and context play an inevitable role in their interpretation of certain human behaviour. This view of multifaceted reality emphasises that the knowledge of human thoughts and behaviour is always laden with subjectivity, and there is no absolute objective conclusion or explanation of human-related phenomena. This epistemological approach of human knowledge is called 'interpretivism' (Bryman, 2012; L. Cohen, Manion, & Morrison, 2011; Paul, 2005), and highlights the fact that social science knowledge is obtained through a unique interpretation by an individual of a particular topic. This approach counters the belief in objective truth of the post-positivist epistemology. Interpretivism, therefore, celebrates diverse subjectivity with a belief that such diversity enriches the understanding of human behaviour (Paul, 2005; Williams, 2000).

Since this research aims to investigate the relationship between the learners' higher-order thinking skill of reasoning with the DGS environment, it focuses on the learner's thinking and learning processes in a particular environment. These abstract cognitive processes need to be studied through the learners' reactions and behaviour in the selected research setting. It also demands the researcher's interpretation of such reactions and behaviour in order to make sense of the studied relationship since these cannot be obtained simply from observation by perception. This leaves room for the interpretation process which plays an essential role in this research. For this reason, the epistemological approach adopted in this research provides a version of an interpretation of the data framed by the researcher's unique background, personality and viewpoint.

Besides subjective interpretation of the obtained data which is the key characteristic of the interpretivist approach to knowledge acquisition, interpretivism also emphasises the importance of the context upon which the interested situation is based. Cohen, et al (2011) and Bryman (2012) claim that an interpretive researcher sets out to understand an individual and his/her interpretation of the world around them. They emphasise that interpretive research is strongly context-based and can never be context-free. The environment of the subject always plays an essential role in his/her reactions and behaviour and cannot be disregarded. This aspect of interpretivism is strongly congenial to the main query of this research where the student's reactions and behaviour in the DGS environment in the Thai context are the main focus.

Williams (2000) also states that an individual is free to attach different meanings to the same actions or circumstances, highlighting the uniqueness and independence of an individual mind. This suggests indeterminateness of the truth or reality where there is no common principle to judge whether an individual's interpretation is right or wrong. Though interpretivism strongly advocates subjectivity over objectivity, interpretive research does not intend to claim the truthfulness of its findings based solely on the researcher's individual subjective interpretation of the data. Interpretive research aims to present just a single interpretation of the research circumstance, and readers or scholars are welcome to interpret the same data independently, giving a different reading of the same circumstance. The dialogue between these different interpreters encourages a process of moderation (ibid) in these interpretations. Interpretive research, therefore, aims to gain inter-subjective interpretation of the circumstance in question by presenting one interpreted version of reality, inviting others to share their different opinions. This epistemology illustrates the notion that the social world can be understood from the collective viewpoints of individuals, each giving subjective interpretations of the phenomenon based on their social backgrounds, beliefs, ideas and mindsets, contributing to the inter-subjective interpretation of the situation (L. Cohen, et al., 2011; Creswell, 2006).

From this characteristic of the interpretive stance, as a researcher, I play an eminent role in how data is obtained and interpreted. The whole conduct of the research and the interpretation of data are inevitably executed through my personal viewpoint, which may be different from others. This means that with the same research aim, research questions, research model and framework, the outcome may be completely different had this research been conducted by another researcher. For this reason, my research is expected to provide just one version of possible answers framed by my own personal experiences and worldview. That said, I attempt to do my best to avoid any bias, and the research is conducted in such a way so as to seek the answers to the research questions neutrally, though I am aware prejudices may still be present. This research is also conducted having regard to other scholars' works in the same field with the aim of contributing my subjective answers to this broad area of study.

5.2 QUALITATIVE CASE STUDY AS THE RESEARCH METHODOLOGY

With the nature of 'what' and 'how' in the main question and sub-questions, this research naturally falls into a qualitative type of research. Qualitative research involves an interpretive, naturalistic approach to the world; studying things in natural settings and making sense of the phenomena in terms of the meanings people give them. Contrary to quantitative research, qualitative research emphasises the qualities of entities, processes and meanings that cannot be scientifically measured (Bogdan & Biklen, 1998; Denzin & Lincoln, 2005; Freebody, 2003).

The information needed to answer this question requires a closer look at the learner's reaction to the given geometric activities in such a technological environment. The research methodology to serve this purpose is a qualitative case study: a study of actions in a real setting in order to understand the situation more clearly (L. Cohen, et al., 2011). A case study is usually a small-scale study aiming to investigate the in-depth mechanisms of a particular phenomenon in a real setting. Though its generalisation may be limited due to the modest scope of the samples, the unique trade-off of a case study is the detailed description of what is occurring in the interested situation.

With the breakdown of this research question into three sub-questions based on the model of study, two separate strategies are needed in order to gain information for all these three sub-questions. The first and the second sub-questions clearly demand a probe into learners' thinking processes: "How do learners use their geometry knowledge to reason how DGS portrays Euclidean geometry?" and "What kind of reasoning strategies do learners adopt in geometric construction and exploration tasks in the DGS environment?". The empirical method to help the researcher gain this insight is the 'task-based interview': a method where the researcher provides tasks for the subject and then persistently questions the subject in order to elicit the learners' cognitive process during the tasks (Flavell, 1963). Besides asking the learners about their strategies during the tasks, the researcher also needs to observe the learners' behaviour during their performance. These observations can provide the non-verbal parts of the learners' reactions and can help in enriching the information needed for these two sub-questions.

The third research sub-question clearly asks for a process of design and justification of the tasks in order to serve the research purpose: "What approach to task design will enable the learner to use reasoning strategies to acquire knowledge in Euclidean geometry?" It involves the stage where the researcher prepares instruments to be used in the task-based interview. This part of the information can, therefore, be gained from the researcher's own thinking process in setting the geometric tasks and is directly presented by the researcher's written explanation.

The basic research methods to be adopted in this research are therefore, interviewing, observation and a description of the researcher's own thinking process in designing tasks.

5.3 RESEARCH METHODS

This section further discusses the two main research methods i.e. the interviewing and the observation. It elaborates a particular type of each method adopted in this research.

5.3.1 Semi-Structured Interview Method

With the focus of this research being the examination of the learners' higher-order thinking skill of reasoning in geometry tasks, semi-structured interviews should provide a suitable strategy for such investigation. A semi-structured interview usually has a clear topic of interest and a list of questions that the interviewer should pursue in order to elicit responses from the interviewee. Nevertheless, the structure and the extent of the questions are flexible, giving the interviewer an opportunity to follow up any unexpected responses or encourage elaboration of the answers through open dialogues (Bryman, 2012; Denscombe, 2010). For this research, the learner's rationale or reasoning for their chosen strategies are set as a topic of interest for the researcher to pursue, and the interview is structured by the given tasks which are designed to challenge the learner's reasoning.

Although learners' reasoning and learning processes are mainly elicited from their verbal responses, other relevant information such as the interviewee's tone of voice, use of language, facial expressions, emotions, etc. should also be looked at in order to assess the interviewee's states of mind at a particular instance. These emotional aspects of their reactions should support the meaning conveyed through verbal dialogues. They should provide another useful part of data to be analysed together with the actual spoken words though these are also subjected to the researcher's interpretation.

Besides the task-based interview, the researcher also arranges a short 'Critical Incident Recall' interview: an interview to capture the thinking processes about a particular incident the subject experienced (Chell, 2004) in the following day after the main interview in the main study phase. For this research, 'Critical Incident Recall' interview is used to elaborate students' reasoning strategies by asking them to further explain their thinking process while viewing the excerpts of video-audio recording of the screen they did the day before. It is expected to help the researcher understands students' strategies better as well as to provide the opportunity to clarify ambiguous reactions the researcher does not manage to pursue during the interview with the students. The 'critical incidents' are identified by the researcher beforehand after reviewing the recording. They are selected based on the distinctiveness of the reasoning processes.

5.3.2 Participant Observation Method

Besides asking the learners for their reasoning strategies through interviewing, their overall behaviours during the tasks can also provide a useful indication of their responses, especially mentally. Observation can be another useful data collection technique to compliment the interview method in this research.

Observation technique offers a direct contact between the observer and the interested situation in a natural setting allowing the observer to experience the circumstance first-hand (Bogdan & Biklen, 1998; L. Cohen, et al., 2011; Denscombe, 2010). However, observation technique generates the interference effect, where the presence of the observer may change the behaviour of the subject, leading to an unnatural situation which can mislead the observer. This problem is impossible to avoid and the researchers should be very discreet about their presence and be clear about their role in the situation.

In order to make the research setting as natural as possible, the observation technique chosen in this research is the participant observation where the researcher plays an active role in the circumstance, not just passively recording the observation (Bogdan & Biklen, 1998; L. Cohen, et al., 2011). The researcher introduces the tasks to the learners and actively encourages them to provide reasoning by challenging their claims. With this participation approach, the subjects' appreciation of the researcher as a passive observer diminishes since he/she also plays the role of a conductor in the activity. The technique of participant observation allows the researcher to interact with the subjects and provides the opportunity to interpret their behaviour more closely. Nevertheless, all these observations are a subjective interpretation of what is occurring and another researcher may have an alternative questioning strategy and a different reading of the subject's reactions.

5.4 RESEARCH PLAN

This section gives a detailed plan of how the research is conducted. It includes the processes and physical settings of the research, as well as the separation of the whole research into three temporal phases.

5.4.1 Empirical Processes

In order to answer the main research question based on three identified research sub-questions, as the researcher, I devise a plan to conduct an empirical work to investigate these questions. The sub-question that needs to be tackled first is the third sub-question which involves the design of the tasks. These tasks are then used as enquiry tools in the other two research sub-questions. The main criterion to justify this third research sub-question is how well these tasks help the researcher to investigate the first and second sub-questions.

The main purpose of this research is to examine how the learners use their geometric knowledge to interpret the DGS environment, and how they reason in the geometric construction and exploration tasks in that environment. The designed tasks aim to cover three key aspects:

I. The student's knowledge of Euclidean geometry

The first part of the tasks is to examine the student's prior knowledge of Euclidean geometry relevant to the intended contents of the task. In the task-based interview, the researcher starts each task by asking students about definitions of the key terms or concepts such as; 'What is an isosceles?'; 'Do you know what the tangent of a circle is?' This is to ascertain the level of knowledge needed for each task. However, if any student is unable to give an explanation of the term or concept, or if any of the students show a lack of understanding, as the researcher, I briefly explain the correct definition or description of the term or concept so that the student can pursue the intended tasks.

II. The student's reaction to the DGS tool

This part of the tasks concentrates on the student's interpretation of the DGS environment and their utilisation scheme of the DGS tool. The key actions involved are the student's construction of pictures or geometric figures in the DGS environment and their uses of DGS commands to explore the geometric situations presented to them. This aspect of the task is intended to investigate the research sub-question 1.

III. The student's reasoning strategies

Another important part of the tasks is the student's reasoning. Students are asked to explain either why their constructions conform to the given instructions, or why geometric figures behave in the way they observe them. This aspect of the task is intended to investigate the research sub-question 2.

The designed tasks are sequenced from the simplest tasks to the more mathematically challenging ones. They start with an introduction to a DGS task, where students are invited to explore the menus of DGS to see what this tool provides. They are then asked to draw any picture they wish using the tool. This task aims to elicit students' reactions to the DGS tool in a non-geometrical situation, especially of how they make sense of DGS commands and the way these commands work. They are also asked to reason why they think DGS works in such a way.

Once the students are sufficiently familiar with the DGS tool, the subsequent tasks involve students' constructions of simple geometric figures with a certain set of DGS commands, as well as their justifications of such constructions. These tasks are intended to examine how students connect their experience with the DGS tool to their prior knowledge of geometric properties, and what strategy they adopt to verify their constructions. The concept of a robust figure which cannot be ‘messed-up’ by dragging are also introduced as a challenge to the students by the researcher. This should lead them to encounter the parent-and-child relationship in the DGS environment. Student’s reactions to this unique DGS constraint are also examined.

In order to enrich the challenges of students’ reasoning strategies, geometric exploration and problem-solving tasks are included in the empirical processes of research. For geometric exploration tasks, students are shown pre-constructed sketches. The researcher instructs them to perform some simple additional constructions, and they are asked to use the DGS tool to explore the figure and report the observation. They are also prompted to explain the reason behind the geometric behaviour they observe. The geometric exploration tasks aim to provide situations where students may adopt abductive reasoning to hypothesise discovered geometric properties, which may lead them to use deductive or inductive reasoning for further verification.

As discussed in Sub-section 2.4.6 in the Literature Review chapter that problem-solving activity can require a range of reasoning strategies from the learners, the designed task to be used in this research, therefore, includes a problem-solving section. For a problem solving task, students are challenged to solve problems involving geometric properties. The task also aims to provide students with situations where abductive reasoning may be adopted. Though the geometric exploration and problem-solving tasks would not involve extensive construction processes, the focus of these tasks is placed on their geometric reasoning using DGS as an exploration tool.

The DGS commands which students are expected to use in these tasks include the drag tool, construction tools and display tools. The measurement and transformation tools (translation, reflection, rotation, dilation), though available in most DGS, are not included in this research. The main rationale for the exclusion of these tools is that this research aims to investigate students’ reasoning in Euclidean geometry where numeric measurements or transformations are irrelevant. Moreover, research by Ruthven, Hennessy et al. (2004) also indicates that measurement anomalies in DGS, which are as a result of number rounding can cause a significant problem when numeric measurements present contradicting data to geometric theorems. This study, therefore, concentrates on tasks based on the tradition of Euclidean geometry without the use of measurement or transformational geometry.

The relationship between the tasks and the research sub-questions are summarised in Table 5.1 below.

Table 5.1 Tasks and research sub-question relationship

Research Sub-questions	Tasks	Data Collected
RQ 1	Task 1: DGS Exploration Task 2: Geometric Construction	Student's interpretation of DGS environment Student's reasoning of their constructions
RQ 2	Task 2: Geometric Construction Task 3: Geometric Exploration Task 4: Geometric Problem	Students' reasoning of their constructions Student's reasoning of their observations Student's reasoning of their solutions
RQ 3	Strategies to design tasks - Student's prior knowledge - Student's reaction to DGS - Student's reasoning	The criterion for task designs

For the first and second sub-questions, task-based interviews are used to pursue the answers with interviewing and observation as basic data collection techniques. As discussed in Sub-section 5.3.2, observation by an individual has a weakness of viewpoint limitation so the technological tool is used to reduce such an obstacle. Besides the first-hand observation by the researcher, recordings from the video and audio capture tool are collected to give the researcher an opportunity to review the interviews after the empirical process. For video capturing, two viewpoints are focused on; the actions on the computer screen and the students' physical reactions during the tasks.

For the screen actions, screen-capturing software is used to record every mouse movement around the screen and keystrokes, as well as the students' synchronous dialogues. For the physical reactions, a video camera is used to film the students and the screen from behind, so their physical actions such as pointing with fingers can be observed. Though it may be useful to put more cameras in place to capture different viewpoints of the students' reactions, such as their facial expressions, the more equipment used in this setting, the stronger the effect of interference on the subjects. With multiple cameras filming them, students may change their behaviour due to the awareness of such exposure. For this reason, it is more appropriate to have just one camera placed out of sight behind the students.

During the task-based interview, the student(s) sit(s) in front of the computer screen and control(s) the DGS with a keyboard and mouse. The researcher sits beside the student(s) to give instruction and ask questions. The physical layout of the task-based interview is shown in Figure 5.1.

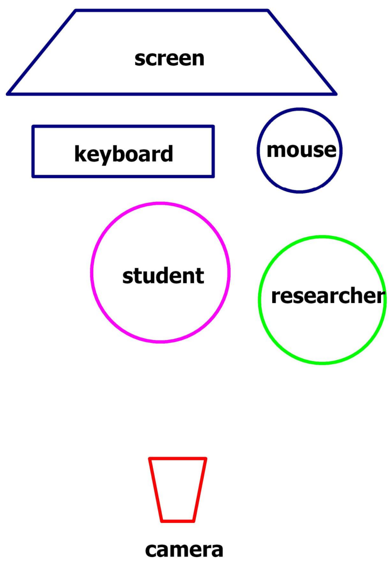


Figure 5.1 Physical layout of the task-based interview

This layout illustrates the research's empirical design that compromises the benefit of recording the circumstances from as many viewpoints as possible, in order to reduce any interference in the setting and to make the students feel more comfortable.

5.4.2 Research Phases

In order to answer the main research question from the three sub-questions, the research is separated into three distinct phases: development of tasks, pilot study and main study.

Phase I: Development of Tasks

This phase concentrates on the design of geometric tasks being used in the subsequent pilot and main studies. The execution of this phase provides the answer to the third research sub-question: “What approach to task design will enable the learner to use reasoning strategies to acquire knowledge in Euclidean geometry?” The main participant in this phase of research is the researcher who is responsible for the development of tasks. The direction of the design process aims to produce tasks which can help the researcher to answer the first and second research sub-questions. These include tasks which provide the learner with the opportunity to interact with various features in the DGS, and also include geometric tasks to encourage the learner’s reasoning processes. The researcher, as the designer, needs to explore a range of commands and features in the DGS, as well as Euclidean geometry rules and properties, before setting a suitable criterion for selection and development into geometric activities.

Phase II: Pilot Study

This phase is set to trial the tasks developed in the first research phase. It is conducted with the students to see how the designed tasks fit the purpose and how they should be adjusted/modified in order to provide answers to the first and second sub-questions. It also gives the researcher guidance on the time-frame needed for each tasks.

Apart from the tasks trial, this pilot study phase also provides the researcher with an opportunity to compare different possible settings for selection in the main study. The students work individually and in pairs during the pilot study in order to examine the benefit and trade-off between these two different settings. While working individually can give the learner complete control of the software, the data collection process needs to rely on the unnatural method of ‘thinking aloud’ which is needed to externalise ‘verbal thought’; a mediator of mental operation according to Vygotsky’s theory. Working in pairs, on the other hand, forces the learner to share the software control thereby encouraging more natural conversation which should reflect their thinking process more faithfully. Furthermore, working in pairs has the advantage of prior knowledge sharing among students, which may improve their performance of the task. However, the reasoning strategies adopted during the tasks are likely to be collaborated which may be more difficult to analyse than in the individual cases. Working in pairs also enriches the dimension of argumentation since the interlocutors are not only between students and the researcher, but also between the students themselves. The experiences gained from this pilot study phase provide the researcher with better insight into the design of the appropriate tasks, as well as the empirical setting best suited in order to elicit the data for the research questions.

Phase III: Main Study

After the tasks and different settings are employed in the pilot study phase, the modified tasks and selected settings are then executed in the main study. The data collection process takes the form of a task-based interview where the researcher presents students with a series of tasks in the DGS environment. Students’ verbal expressions, either in the form of ‘thinking aloud’ or in actual conversation, provide the key data source for their reasoning. However, their overall behaviour can also contribute to supporting the data analysis and this is gained from the observation technique.

5.4.3 Role of the Researcher

Since this research aims to investigate students' higher-ordering skill of reasoning in the dynamic geometry environment through geometric tasks, it has no pedagogical aim for the students to achieve any learning objectives. What interests the researcher is how students react to the tasks in this environment rather than whether or not the students actually learn the relevant geometric concept. All the tasks are therefore designed in order to help the researcher gain such an insight into the students' thinking without the objective of educating the participant students. The inquisitive approach to students' thinking reinforces my role as a researcher. This role should be clearly distinguished from that of a teacher where tasks are designed to help students learn new concepts.

Besides being the researcher, I also play the role of a reluctant believer during the interview sessions, challenging students for better reasoning. This particular dimension of the researcher role is integrated into the designed tasks where an individual is acting as an interlocutor providing live interaction to the students to stimulate the reasoning activity. Nevertheless, the role of a reluctant believer should give complete freedom to the students to think for themselves. The reluctant believer does not in any way lead or guide the students to a particular thinking path in order to help them learn the intended property as most teachers would.

In order to avoid any possible impression that I am a teacher, I was introduced to the students by the teacher as her ex-pupil from the school and now a PhD student returning to conduct research in particular geometry software. The students therefore saw me as their school senior and they called me 'big brother' instead of 'teacher'. This helped them to be aware that all these activities were not part of their classroom education but just a part of the educational research for my PhD.

5.5 RESEARCH SETTING

Once the empirical process structure has been laid out, the actual research setting is then considered and decided. This section discusses the criteria for choosing the school and the participant students for each phase of this research.

5.5.1 Choice of School

Apart from Thai nationality, this research places no emphasis on any particular characteristic of the learners taking part in the research. The subject samples can, therefore, be generic in order to improve the generalisibility and all extreme or unusual cases should be avoided. This also leaves room for convenience sampling (L. Cohen, et al., 2011) where the subjects can be selected based on the researcher's ease of access.

In order to avoid a possible cultural gap between myself as the researcher and the subjects, this research is conducted in a cultural setting familiar to me, i.e. in the same school I once attended. This helps me to relate to the cultural background of the chosen subjects, including the environment and the use of language, especially the regional dialect. The school is approached via IPST, Thailand, the sponsor of this research through a teacher who has already worked with IPST.

5.5.2 Choice of Students

The age range of the subject students is set at 14-15 years old. They should be in the second year of lower secondary education. The rationales for choosing students of these ages are:

1) Thai students at this age should be able to use basic computer hardware and software with mouse and keyboard and be familiar with standard operational and application software features such as managing files, accessing menus or using undo/redo commands.

2) Students in their third year of lower secondary education should have the necessary basic geometric knowledge to undertake a wider range of activities in the research tasks.

Since this research aims to study the fresh reactions of students to the DGS environment, all the selected students should have no prior experience using GSP or any other DGS product. There are six students in the pilot study phase and eighteen students in the main study phase. These are the maximum number of students I am able to manage in accordance with the research time-frame.

The process of student selection is aimed at gaining diverse strategies and responses to the designed tasks, and therefore the sample should include both boys and girls with varying degrees of mathematical ability, i.e. above-average, average and below-average. The numbers of boys and girls are balanced to provide the most representative sample under the confined scope of this qualitative case study, and to avoid a bias from purposively selecting a particular group of students to take part in the research.

In order to select students who agree to participate in the research, the teacher first separates the boys and girls in the two classes she is in charge of and then categorises each student into above-average, average and below-average groups. Since the task-based interview is planned for 70 minutes, the interview needs to be conducted during the students' free time. Fortunately, students in these two classes are given three free-period slots in each week and they are supposed to enrol in one elective course and leave the other two slots free. It is therefore possible to conduct research during this free-period every day, depending on which students are free on that day. Based on the list of boys and girls of varying mathematical ability, and the free schedule of the students, the researcher randomly selects the available students for the interview. During the pilot study, three boys and three girls are selected to take part in the research. One boy and one girl are interviewed individually, while the two boys and two girls work in pairs. This is to explore different research settings in order to organise the final setting for the main study, yet to be decided. In any event, nine boys and nine girls of varying abilities are interviewed in the main study.

5.6 ETHICAL APPROACH

In order to ascertain the ethics of this research project, the ethical guidelines provided by BERA (British Educational Research Association, 2004) are strictly followed. The school and the participant teacher are contacted via my sponsor, IPST, who accepts all responsibility for my involvement with the school, teacher and students. As for the students, they are given an information sheet informing them of all the details of this research, including the topic, objective, and data collection process as well as the contact details of the researcher, supervisor and King's College London. The students and their guardians are also given consent forms which they are completely free to agree or reject in order to join in the research. Fortunately, the students and their guardians were very eager to be a part of this research, and all the students in the two classes where the participant teacher is in charge, agreed to participate. During the data analysis and presentation, the real names of the participant students are not revealed and they are referred to by pseudonyms. All the raw data is destroyed after the completion of the research. The documents pertaining to the ethical approach of this research are shown in Appendix A.

5.7 EXPECTED DATA AND ANALYSIS STRATEGY

After the empirical phase of this research, two main parts of data are collected. The first part is the researcher's rationale for developing the tasks designed for the task-based interview; both in the pilot study and main study phases. This data should provide answers to the third research sub-question: "What approach to task design will enable the learner to use reasoning strategies to acquire knowledge in Euclidean geometry?"

For the second part of the data gained from the task-based interview: the video and audio recordings from the screen; the physical reactions of the students from behind; the GSP files of the task performed; and the researcher's observation notes, provide raw data for the research analysis phase. The main strategy used to analyse this data is axial coding; a qualitative coding technique with various related codes into categories of common meaning shared by the group of codes (L. Cohen, et al., 2011). Axial coding strategy allows the researcher to analyse the data, based on the relevant concepts raised by other scholars or researchers in past research in the same field. It is expected to direct the analysis to relate to other studies as much as possible, while leaving room for new issues that may arise during the course of analysis. This is opposed to open coding where the codes emerge from the data itself (ibid.).

The codes used in axial coding analysis are categorised based on topics from the Literature Review chapter. The categorisation and the description of criterion for each key code are given in Table 5.2.

Table 5.2 List of axial codes for data analysis

Axial Codes	Criteria
Reasoning	Response to the question 'why?'
Inductive Reasoning	Reasoning by inferring to observation of the case(s)
Transformational Reasoning	Reasoning involving a dynamic process
Deductive Reasoning	Reasoning by inferring to known rules or definitions with confidence
Abductive Reasoning	Reasoning by posing a possible hypothesis to explain
Toulmin's Model	
Data	Evidence learners use for reasoning
Warrant	Principle used for the justification process
Claim	The statement to be justified
Argumentation	Reasoning in interaction with other(s)
DGS Features	
Drag-mode	The use of the Arrow tool to move the constructed figure
Parent-and-Child Relationship	The dependency of movement or existence of the figure's components

Axial Codes	Criteria
Edit commands	The use of commands in the Edit menu
Display commands	The use of commands in the Display menu
Construction commands	The use of commands in the Construct menu

These axial codes are elaborated and exemplified as follows:

a) Reasoning is identified as the students' verbal response to the question 'why?' or to the challenges placed by the researcher in order to make a conjecture or to explain the observed phenomena. Though reasoning may originally be cognitive, the reasoning in this research is identified through verbal responses only. Mere actions in response to the researcher's prompt or challenge is not considered as primary reasoning data in this research. 'Reasoning' in the data analysis phase is therefore concentrated on an 'interpretable' verbal response. The axial code 'reasoning' can be logical or illogical, valid or invalid and relevant or irrelevant to the interested topic.

- Inductive Reasoning is identified either by the process of generalising common properties among two or more individual cases or the process of falsification from one or more cases to negate the statement in question. Inductive reasoning is usually based on observation, where a common property is visually spotted across multiple cases. If students reach a conclusion that the midpoint theorem is true because it works with the three different triangles they observe, this would be considered inductive reasoning based on their visual observation using multiple cases to confirm the validity of the statement. On the other hand, showing that one single case violates the conjectured property would also be deemed as inductive reasoning by falsification, since the main strategy is to use the observed circumstance to verify the falsity of the assumed property.

- Transformational Reasoning adopts a similar approach to Inductive Reasoning with the key difference being that a generalisation is made from a dynamic change of the case rather than multiple individual cases. It identifies the invariant and variant properties under change, leading to an explanation using a mechanism of the movements of the observed figure. If students verify a certain property with a reason that such property is unchanged under dragging in the DGS, this would be interpreted as transformational reasoning. If students try to verify the invariant property with an explanation based on the mechanism of change, such as: if this angle is getting bigger, this angle must be smaller, keeping this angle constant, this would also be identified as transformational reasoning since it is based on observation under a dynamic circumstance.

- Deductive Reasoning is identified by the logical reference to the known rule or property as an explanation to the observed property. The axial code 'Deductive Reasoning' in this research is identified through the confident reference to a certain property whether it is true or not. Reasoning can still be deductive even if it is false. For example, if a student refers to the property of similar triangles to verify the parallel property of the triangle's midpoint theorem, this would be deemed deductive reasoning.

- Abductive Reasoning is identified in this research as the students' formation of explanatory hypothesis in order to explain the observed phenomena. Such hypothesis may or may not in the end successfully verify the circumstance, but the fact that students attempt to pursue such hypothesis as the plausible explanation is sufficient to categorise it as abductive reasoning. If a student makes a guess that a certain rule or property might be a possible explanation for the observed phenomena, this would be interpreted as abductive reasoning, since explanatory hypothesis is already formulated, even though such hypothesis may fail to finally verify the conjecture. The key difference between abductive reasoning and deductive reasoning in terms of rule or property reference is the level of confidence of the reasoner towards such rule or property. If the student feels that the identified hypothesis 'might' explain the observed property, this would be categorised as abductive reasoning, but if they feel that such rule or property is the prime reason for the observed figure's behaviour, it would be categorised as deductive reasoning.

b) Toulmin's Model is identified through the following three axial codes:

- Data in Toulmin's model of argumentation is identified as any object that the reasoner uses as evidence to support the statement in question. It can be an observation of a single geometric figure, identification of the perseverance of a certain property or even a known rule or prior knowledge used to validate the statement.

- Warrant in Toulmin's model is identified as a personal way of using selected data to secure knowledge. Despite the term's implication, 'warrant' does not necessarily 'confirm' that the conclusion is mathematically valid. This axial code is treated rather as an individual approach of knowledge confirmation. It can be the deductive reasoning process of formal proof, empirical justification, intuitive visualisation or even the testimony of trusted others.

- Claim is identified as a statement the reasoner wishes to verify with the 'data' through the process of 'warrant'. It can be in the form of conjecture, hypothesis or the tentatively invariant property.

c) Argumentation is identified as reasoning beyond an individual cognition extending to the debate, discussion, information sharing and reconciliation of a certain claim with another. Despite the confrontational implication of the term, argumentation can either be agreeable or disagreeable. The main criterion for the presence of argumentation is that each party should have an opportunity to have a say about their idea or belief whether such idea or belief is agreeable to their counterpart or not.

d) DGS Features are identified through the use of the following GSP functions:

- Drag-mode is identified as the attempt to use the Selection Arrow tool to move the figure. Note that the axial code 'Drag-mode' used in this research refers to the use of the plain or Translate Selection Arrow Tool only. The use of the Rotate and Dilate Selection Arrow Tool is excluded from this research. It should be emphasised that the use of drag-mode in this research refers to the 'attempt' to move the figure rather than actually moving it. If the user tries to use the Translate Selection Arrow Tool to move the figure, but fails to do so due to the parent-and-child relationship, this would still be considered as using the 'drag-mode'. This is because an attempt to manipulate the figure is made with this particular tool though other constraints have limited its use.

- Parent-and-Child relationship is identified as the users’ perception that the movements of the constructed figure depend on each other. The key criterion for the identification of the parent-and-child relationship is the ‘dependency’ of the movements, regardless of the correctness of the direction of such dependency.

- Edit commands include the use of Undo, Redo, Cut, Copy, Paste and Clear in the Edit menu in the GSP only, regardless of whether the user understands the functions of these commands or not.

- Display commands include the use of Line, Width, Colour, Hide/Show objects and Show Labels only, regardless of whether the user understands the functions of these commands or not.

- Construction commands include the use of Point on Object, Midpoint, Intersection, Parallel Line and Perpendicular Line Construction only, regardless of whether the user understands the functions of these commands or not.

5.8 VALIDITY AND TRUSTWORTHINESS

One of the factors commonly used to assess educational research is validity, i.e. how the research instrument actually elicits the knowledge of such research claims (L. Cohen, et al., 2011; Gay, Mills, & Airasian, 2006; Golafshani, 2003). The validity of the research helps to ascertain that its findings conform to what actually happens in the circumstance, and satisfies its primary rationale to provide a valid explanation of the interested phenomena. This research is designed to strengthen its validity. Attempts have been made to ensure that the outcomes genuinely represent the relationship between the Thai lower-secondary students' higher-order thinking of reasoning, and the dynamic geometry environment, which is the main focus of this research. Literature regarding higher-order thinking, reasoning, argumentation, the geometry domain of mathematics, geometry education and the dynamic geometry environment has been thoroughly studied in order to formulate a clear definition of the characteristics of these concepts. As the research instrument, the interview tasks are carefully designed to challenge students' reasoning, together with the persistent question 'why?' by the researcher in the GSP environment. These tasks are reviewed and revised after the pilot study in order to ascertain their capability of initiating the students' reasoning process. The tasks are also designed to be open, giving students complete freedom to reason in their own way, without any lead or guidance from the researcher, apart from the challenge. The overall setting of this research should, therefore, reinforce its validity where the data obtained should demonstrate the relationship between students' geometric reasoning in the DGS environment.

With interpretivism as the epistemological stance of this research, an issue may arise concerning the validity of one point of view in these circumstances by a single researcher. As discussed earlier in this chapter, this research has the modest aim of presenting just a subjective view of the literature and data gained from the empirical phase. It is a tentative interpretation, inviting others to share their ideas for future moderation. This research, therefore, limits itself to the validity of this single viewpoint only. It does not intend to claim the validity of the findings as definitive answers to the set of research questions.

This individual scope of the research interpretation leads to another issue about the trustworthiness of the research or how it is faithful to actuality without any kind of manipulation to deviate from the data (Rolfe, 2006; Shenton, 2004). The trustworthiness of this research can be considered by the two main parties: the participant students and the researcher. In order to enhance the research's trustworthiness, it is essential that the participant students react to the interview tasks naturally without deviant pressure. To achieve this, the researcher attempts to put the participants at ease and inform them that their performance will not be assessed or evaluated. They are free to perform the task in their own way and there is no penalty if they fail to complete it. This strategy should lead to bona fide responses from the students, preventing their tendency to try to impress their teacher, as is usually the case in most classroom activities.

As to the fidelity of the research, it is difficult to verify this myself as I also conducted it. The only way to ensure this is to automatically guarantee that this research is conducted purely by focusing on the research question without any other strategic agenda from any other party, not even the sponsor of this research. The research is by no means influenced by the government's national policy to promote the use of the GSP in Thailand. It is conducted with a genuine interest in investigating the relationship between students' geometric reasoning and a certain dynamic geometry environment. The data analysis and research findings, therefore, stem completely from the researcher's honest interpretation without any interference, providing a genuine individual point of view of the circumstance.

The chosen research method and design discussed in this chapter is considered to be the most appropriate way of pursuing the research question posed in the previous chapter. Though this method and design does have limitations, it should provide a sufficient insight into the study.

6 DEVELOPING ANALYSIS MODEL

After the tasks design, pilot study and main study phases, as the researcher, I gain further insight into the four entities, and the relationship between them in the model of study given in Figure 3.5. Such insight identifies significant issues pertaining to all three research sub-questions. These issues are mapped to the model of study given in Figure 3.5 in order to elicit the possible relationships between them. The four main entities, as well as the relationships between them, are reinterpreted in the light of the data collection and design processes. This provides the base model for the data analysis phase in order to answer the main research question.

The re-interpretation of the model of the study is shown in Figure 6.1.

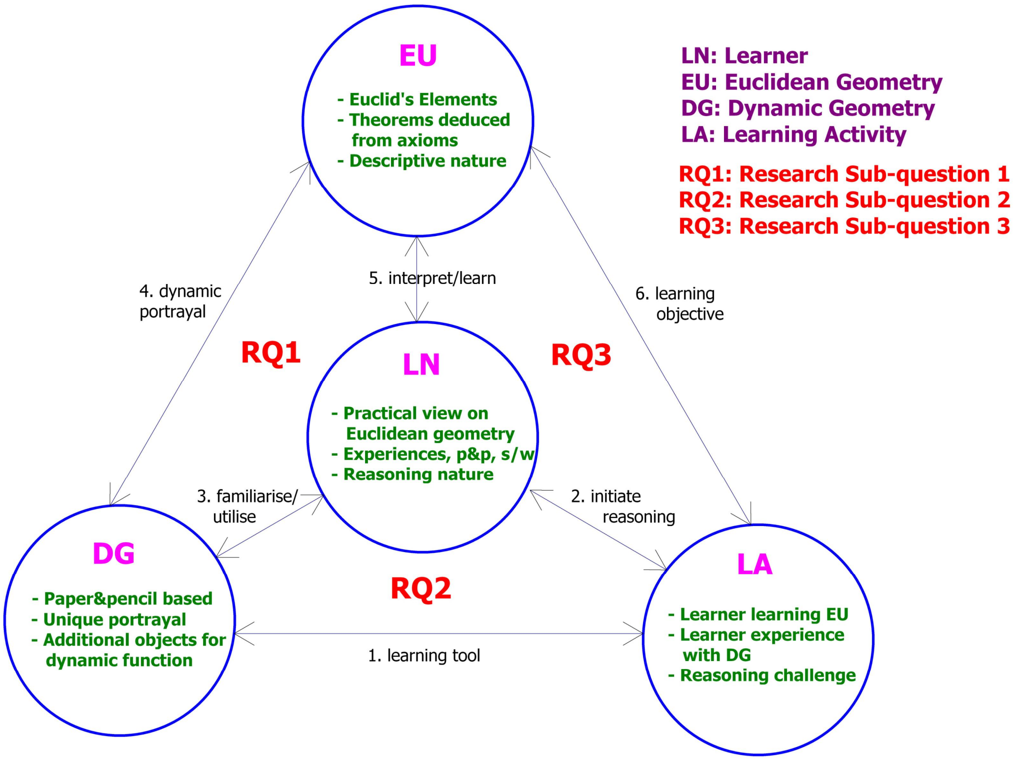


Figure 6.1 Reinterpretation of the model of study

6.1 REINTERPRETATION OF THE ENTITIES

In the light of the tasks design and data collection processes, the broad views of the four main entities identified in Chapter 3 are refined by the important issues found during the empirical phase. This requires reinterpretation of these entities in order to clarify the relationships between them. The reinterpretations of these four entities are discussed as follows.

6.1.1 Euclidean Geometry (EU)

The concept of Euclidean geometry is traditionally based on the compilation book *Elements* which is constructed as a series of deductive-axiomatic statements based on a set of agreed axioms or postulates. The nature of Euclidean geometry is descriptive and based primarily on the deductive way of reasoning. Nevertheless, Euclidean geometry is usually taught in schools in the paper-and-pencil environment. The diagrams in the paper-and-pencil tradition provide graphic representations of the Euclidean geometry concept. Students, therefore, learn the rules of Euclidean geometry through the properties of concrete shapes in the paper-and-pencil environment. The different nature of concrete graphic representation of the paper-and-pencil environment and the abstract deductive rules of Euclidean geometry lead to tension when students need to interpret Euclidean geometry through concrete diagrams. Sub-section 2.6.3 in the Literature Review chapter discusses a suggestion by Laborde (1993) that drawing in the paper-and-pencil environment has the dual role of a concrete visible entity and a theoretical object. This generates a gap in the relationship that students need to bridge in order to understand the Euclidean geometry rule from the diagram. Moreover, there is also a representational gap in the role of the theoretical object of the drawing since the concrete drawing can never be completely faithful to the theoretical object, e.g. it is impossible to draw a widthless line in the paper-and-pencil environment. The concrete drawing of a diagram, therefore, has a crucial limitation in representing theoretical Euclidean geometry. Such a gap can be a huge challenge for students to overcome. The ideality of Euclidean geometry is usually hard to achieve for these reasons.

6.1.2 Learner (LN)

Since the learners in this research experience Euclidean geometry through the diagrammatic paper-and-pencil environment, it appears that they have different views of what Euclidean geometry is from the mathematicians' viewpoint. Students usually have a very practical view of Euclidean geometry. They consider geometric shapes to be basic forms which can be used to construct a meaningful figure in order to design something useful in a real life situation. The abstract concept of Euclidean geometry is impractical and pointless for them. Judging from the students' performance, it seems that most of them are unfamiliar with the deductive proof process. The concept of Euclidean geometry deduction is rather alien to them. This provides tension between the students' nature of inductive reasoning based on a visible concrete diagram, and the deductive-axiomatic structure of Euclidean geometry, highlighting the gap mentioned in the previous sub-section.

The learners' prior experience both in their geometry knowledge, such as definitions and rules, as well as their knowledge of computer use also play a significant role when they try to make sense of how the DGS tool works. These prior experiences can both help and disrupt the process of software familiarisation.

The learners, therefore, participate in this research with their existing knowledge and experience which significantly affects the way they perform the task. They also possess a reasoning nature making them inclined to opt for a particular type of justification. The learner's reaction to the task-based interview and challenge from the researcher in the DGS environment may help transform such experience or nature. The change of their understanding or reasoning nature from this circumstance and environment can be viewed as the Zone of Proximal Development (ZPD) proposed by Vygotsky (1978) discussed in Section 2.5. This change may provide evidence of the transition in the lower and higher mental skills of reasoning.

6.1.3 Dynamic Geometry (DG)

In essence, Dynamic Geometry Software is based on the paper-and-pencil tradition of geometry learning. It portrays Euclidean geometry based on diagrams rather than axioms as in Euclid's *Elements*. The reasoning nature of the DGS environment is therefore inductive rather than deductive since visible diagrams lend themselves to perception-based reasoning. This tendency again provides another tension when reasoning in the DGS and Euclidean geometry environment. What's more, DG also reinterprets the definition of geometric shapes in order to present them more accurately in the computer environment which differs from the traditional paper-and-pencil mode. An example of this is the portrayal of rays or straight-lines which are extended beyond the screen and can be further explored through scroll-bars.

This varying interpretation can also mislead students who are more familiar with the paper-and-pencil environment. A number of additional objects and features are also needed in the DG environment to facilitate dynamic manipulation. For example, visible end points of a segment are needed in order to modify its length or orientation. These extra elements need to be appreciated and understood by the learners in order to make sense of how the software works. Nevertheless, the dynamic flexibility in DG can be helpful for the students to explore the geometric relationship of the figures in order to discover and examine governed geometric rules.

6.1.4 Learning Activities (LA)

Since the original aim of the designed activities is to let students learn the Euclidean geometry concept using the DGS tool through the process of reasoning, the main raison d'etre for the designed task is to identify the Euclidean geometry topic and the process which students need to go through in order to learn it. Due to the limitation of the research timeframe, selective commands in DGS are introduced to the students. These can guide or limit the students' use of the DGS tool to perform the tasks. The tasks are designed to give students freedom to reason in their own way, but the researcher may challenge them to more rigorous reasoning. This challenge inevitably favours the deductive way of reasoning which leads to universally accepted postulates in the Euclidean geometry domain. The direction towards deductive reasoning can provide another tension since it may not conform to the students' natural tendency to inductive reasoning, based on the given diagrammatic environment. The way the tasks are designed in this research, therefore, can inherently limit the scope of the study and value one type of reasoning over another.

6.2 REINTERPRETATION OF THE RELATIONSHIPS

After the four main entities are reinterpreted based on the research empirical phase, the essential relationships among them are also re-examined to reflect what occurs during the process of data collection. The two-way relationships between each entity presented in the model of study in Figure 3.5 are now reinterpreted combining the two one-way relationships into one two-way relationship with the most essential and pertinent relationship(s) identified and relabelled. The re-thinking of six main relationships between the four entities is already depicted in Figure 6.1 above. They are numbered and are discussed as follows.

6.2.1 LEARNING TOOL (1-LT) Relationship

The Learning Tool (LT) relationship is between LA and DG. From the 'utilises' and 'provides features' between LA and DGS, the key relationship between LA and DG is the adoption of DG as a learning tool for the designed task. With the aim of this research being to investigate the learner's higher-order thinking skill of reasoning in the Dynamic Geometry environment, it is inevitable that Dynamic Geometry is used as the learning tool for the designed tasks. It should be noted that other learning environments such as paper-and-pencil or different graphical software can also be used as learning tools for the designed task. However, the fact that Dynamic Geometry is chosen as the learning tool, frames the strategies the researcher may use in the design process in a particular way. A different learning tool or environment could result in an alternative different design strategy. All the designed tasks, therefore, must have the ability to be tackled in the Dynamic Geometry environment.

The DG tool provides various features which the learner may use for their reasoning process but some of them may not directly relate to Euclidean geometry e.g. transformational geometry and the coordinated geometry features. For this reason, the researcher needs to identify which commands in DG are introduced to the students during the tasks. This also frames the learner's adoption of the DG tool for their reasoning since the learner has a limited choice. Moreover the way the DGS organises its commands can also imply the use of each command, especially when they are grouped together in the same menu. All these factors can initially affect the learner's reasoning strategies.

6.2.2 INITIATE REASONING (2-IR) Relationship

The Initiate Reasoning (IR) relationship is the relationship between LN and LA. From the 'challenges' and 'sets strategies' relationships between LN and LA, the eminent function of the LA to initiate the learner's reasoning activity is identified as the main relationship. The tasks are designed to challenge the learners with problems they have never encountered before. The problems require a reasoning process from the learners in order to validate their observation or construction. Ideally, the tasks should be open to, or cover, all basic types of reasoning: inductive reasoning, deductive reasoning and abuductive reasoning. This is to ensure that the learners' transition from lower to higher reasoning is possible and can be observed. The inclusion of the deductive reasoning possibility in the designed task also provides the researcher with an opportunity to challenge the learners to more rigorous reasoning, though they may not achieve the level. The actual learning activities consist of two main parts: the tasks, and the challenges and reactions placed on the learners by the researcher.

6.2.3 FAMILIARISE AND UTILISE (3-FU) Relationship

The Familiarise and Utilise (FU) relationship is the relationship between LN and DG. Since the research is also intended to investigate the learners' interpretation of the Dynamic Geometry environment, all students participating in this research should have no prior experience of any kind of DG. This allows the researcher to study the students' initial reactions when they first encounter the software and have to make sense of how it works. Nevertheless, the students' prior experience with other computer software is expected to help them learn the command syntax in the DGS environment. The main activities between LN and DG are, therefore, getting familiar with the software, realising what it can do and how to control things, then utilising these features as a tool for their reasoning process during the interview tasks. The learners' selections and utilisation schemes of available DG features to help them reason provides a key source of information when answering the research question.

6.2.4 DYNAMIC PORTRAYAL (4-DP) Relationship

The Dynamic Portrayal (DP) relationship is that between DG and EU. From the 'controls' and 'symbolises' relationships, these can be interpreted as DG's attempt to portray EU in the dynamic mode. DG symbolises EU through the tradition of concrete diagrams in the paper-and-pencil environment. While the rules of EU control how diagrams in DG should behave so they still respect the concept, the fact that DG is actually based on the paper-and-pencil tradition of EU portrayal as well as the gap between the concrete entity and the theoretical entity of the diagrams as discussed in Sub-section 6.1.1 remain in DG's portrayal of EU. Learners, therefore, need to go through this gap when they use DG as a tool to learn Euclidean geometry through reasoning. The graphical nature of DG's visual-based representation together with the figures' behaviour under the dynamic drag-mode concretely portrays the abstract relationship between the properties of the shapes described in the Euclidean geometry concept. The learners need to map these together in order to elicit the intended deductive-axiomatic nature of Euclidean geometry. Besides this obvious gap, DG also incorporates its own set of rules in order to portray Euclidean geometry in the dynamic mode. Its unique governing procedures such as sequential construction and the parent-and-child relationship can also affect the learner's interpretation of the given tasks. This complicates the situation since students encounter both geometric rules and software rules in the same environment.

6.2.5 INTERPRET AND LEARN (5-IL) Relationship

The Interpret and Learn (IL) relationship is the relationship between LN and EU. From the 'exposes rules' and 'learns' relationship, the most significant activities between the learners and EU are the learner's interpretation and learning of EU concepts. This relationship is separated into the two basic activities; interpreting and learning. All the participating students in this research have prior experience of Euclidean geometry through past lessons in the paper-and-pencil environment. This factor plays an essential role in how students interpret the Euclidean geometry portrayed in DGS. Nevertheless, the students appear to have their own view of what Euclidean geometry is. The relationship is, therefore, defined as 'interpretation' rather than 'understanding'. This is to emphasise that students may have their own interpretation of EU which may differ from the traditional understanding of what EU is. From the challenge placed by the designed tasks, students are expected to learn new EU content based on their prior knowledge and reasoning process. The learners' conviction of the EU rule can be a result of any type of reasoning process. Nevertheless, the learning part of this relationship is tentative since the learners may or may not learn new aspects of EU during the task-based interview. In the event that students do not learn any new EU content, they can still gain some experience relating to the EU concept by being exposed to the tasks.

6.2.6 LEARNING OBJECTIVE (6-LO) Relationship

Learning Objective (LO) is the relationship between LN and EU. From the 'frames' and 'represents' relationship, the ultimate relationship between LN and EU is identified as the action to achieve the goal, i.e. LA is designed with the EU concept set as a learning objective. However, the design of LA used in this research has a dual role; as a research instrument and a pedagogical lesson. Research-wise the tasks are aimed to elicit the learner's reasoning process in the DG environment while the pedagogical aim is the acquisition of EU knowledge through reasoning. With EU set as a learning objective of the designed task, it is inevitable that the traditional deductive axiomatic approach of EU is also the preferred reasoning approach. This necessity places a value on one particular type of reasoning, while students may have a tendency to reason in other ways in such an environment. In order to deal with this tension, the task is open to all types of reasoning but the challenge from the researcher is to orient towards deductive reasoning which is the generally accepted means of validating EU knowledge.

The reinterpretations of these six relationships emphasise the aspect of relationships that influence the interaction between these four entities. These provide a basic model for the analysis phase, where such interaction is examined in detail in order to elicit the answer to the research question. The numbered relationships given in Figure 6.1 can be mapped to the research sub-questions identified in Chapter 4 in Table 6.1.

Table 6.1 Relationships and research sub-questions

Research Sub-Questions	Relationships
RQ1 (LN-EU-DG)	3 Familiarise/Utilise (FU) 4 Dynamic Portrayal (DP) 5 Interpret/Learn (IL)
RQ2 (LN-LA-DG)	1 Learning Tool (LT) 2 Initiate Reasoning (IR) 3 Familiarise/Utilise (FU)
RQ3 (LN-EU-LA)	2 Initiate Reasoning (IR) 5 Interpret/Learn (IL) 6 Learning Objective (LO)

6.3 INTER-RELATIONSHIPS OF THE MODEL

Referring back to the model of study (Figure 3.5) developed in Section 3.3, it shows that the indirect relationship between the Learner and Learning Activity mediated through the Dynamic Geometry tool is not yet included. This section concentrates on such mediated relationships in the model. It covers not just the mediated relationship between the Learner and Learning Activity through Dynamic Geometry, but all the mediated relationships among the four entities called inter-relationships. These inter-relationships combine the triangular mapping of all six relationships identified in the previous section. In order to illustrate these inter-relationships, the two-way arrows depicting the relationship between entities in Figure 6.1 are transformed into rectangle boxes with the key issues of each relationship outlined. These relationships are then depicted by new two-way arrows generating fifteen inter-relationships as shown in Figure 6.2. Since the Interpret and Learn relationship is the central activity this research aims to examine, it is placed at the centre of the inter-connected relationships, resulting in a pentagon model.

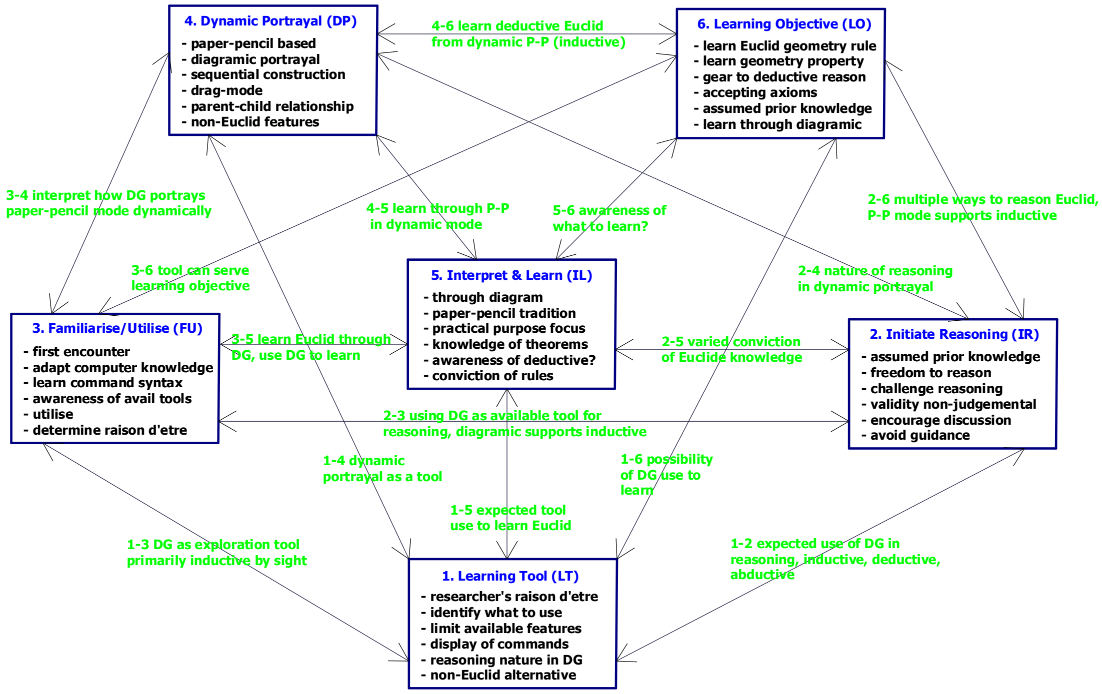


Figure 6.2 Model of inter-relationship

This model presents ten different situations assumed to take place during the research. These assumptions are represented by different triangles portrayed in the model at Figure 6.2. Since the central relationship of interest is the Interpret and Learn relationship, only the inter-relationships concerning this relationship (no. 5) are considered. The ten assumptions based on these inter-relationships are outlined in Table 6.2.

Table 6.2 Inter-relationship and assumptions

INTER-RELATIONSHIPS	ASSUMPTIONS
1-2-5	DG tool lends itself to various reasoning strategies to learn Euclidean geometry including deductive reasoning.
1-3-5	Learner can learn Euclidean geometry through DG tool especially from identified commands.
1-4-5	Learner can make sense of dynamic portrayal in DG and can learn Euclidean geometry through it.
1-5-6	DG can be used as a tool to achieve the learning objective where learner learns Euclidean geometry concept.
2-3-5	Learners may use DG features to help them reason better especially in deductive reasoning.
2-4-5	Learners may utilise dynamic portrayal in DG to help them reason better especially in deductive reasoning.
2-5-6	Challenging learners' reasoning helps them to learn Euclidean geometry concepts.
3-4-5	Learners may use DG features incorporating dynamic functions to learn Euclidean geometry.
3-5-6	Learners are aware that DG tool can help them learn Euclidean geometry.
4-5-6	Learners are aware that dynamic features in DG tool can help them to learn Euclidean geometry.

From these ten inter-relationships, it can be seen that inter-relationships 1-2-5 and 1-5-6 are actually the core hypothesis this research attempts to pursue while inter-relationships 2-4-5 and 4-5-6 can be seen through inter-relationships 2-3-5 and 3-5-6 respectively. This is because the learners view the functions of Dynamic Portrayal through the Familiarise/Utilise relationship. These four inter-relationships, therefore, can be excluded in the data analysis model. The model of study is then modified as shown in Figure 6.3.

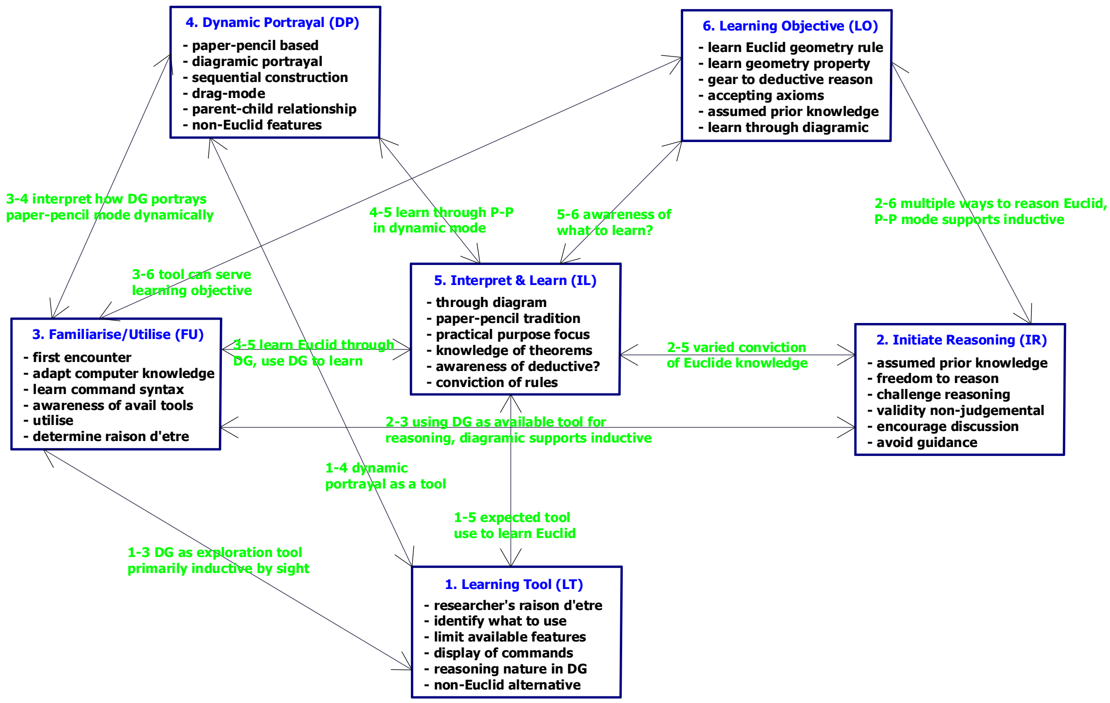


Figure 6.3 Modified model of inter-relationships

The remaining six inter-relationships for the focus of this research are shown in Table 6.3

:

Table 6.3 Inter-relationships for the data analysis model

INTER-RELATIONSHIPS	ASSUMPTIONS
1-3-5	Learner can learn Euclidean geometry through DG tool especially from identified commands.
1-4-5	Learner can make sense of dynamic portrayal in DG and can learn Euclidean geometry through it.
2-3-5	Learners may use DG features to help them reason better especially in deductive reasoning.
2-5-6	Challenging learners' reasoning helps them learn Euclidean geometry concepts.
3-4-5	Learners may use DG features incorporating dynamic functions to learn Euclidean geometry.
3-5-6	Learners are aware that DG tool can help them learn Euclidean geometry.

The inter-relationship 3-4-5 is the basis for the research sub-question 1. The inter-relationships 1-3-5, 1-4-5 together with 2-3-5 are the basis for the research sub-question 2 and the inter-relationship 2-5-6 is the basis for the research sub-question 3 under the influence of other inter-relationships in the model.

This final model is used as the core structure for the data analysis process by applying the data obtained from the empirical phase. The analyses are discussed in Chapters 7-10 with the aim of investigating the inter-relationships identified in this chapter.

7 DESIGN OF LEARNING TASKS

This chapter discusses the overall process of the development of task-based interview activities. It directly involves the research sub-question 3: “What approach to task design will enable the learner to use reasoning strategies to acquire knowledge in Euclidean geometry?” The selected development criteria provide the answer to this sub-question. This chapter covers both the Development of Activities and Pilot Study phases of the research. Though some preliminary data analysis is done during the pilot study, its main purpose is to trial the task design for the main study phase. The design process includes the consideration of different approaches to task design, the identification of the objective of each task, the rationale for choosing the tasks and the possible outcome or performance of the participant students.

The process of task design involves the inter-relationship 2-5-6 presented in the analysis model developed in the previous chapter. This inter-relationship concerns the development of the tasks to initiate and challenge the learner’s reasoning process in the DG environment with the Euclidean geometry’s tradition of deductive reasoning set as the pedagogical objective. The inter-relationship is shown in Figure 7.1, with thick arrows bordering the highlighted area.

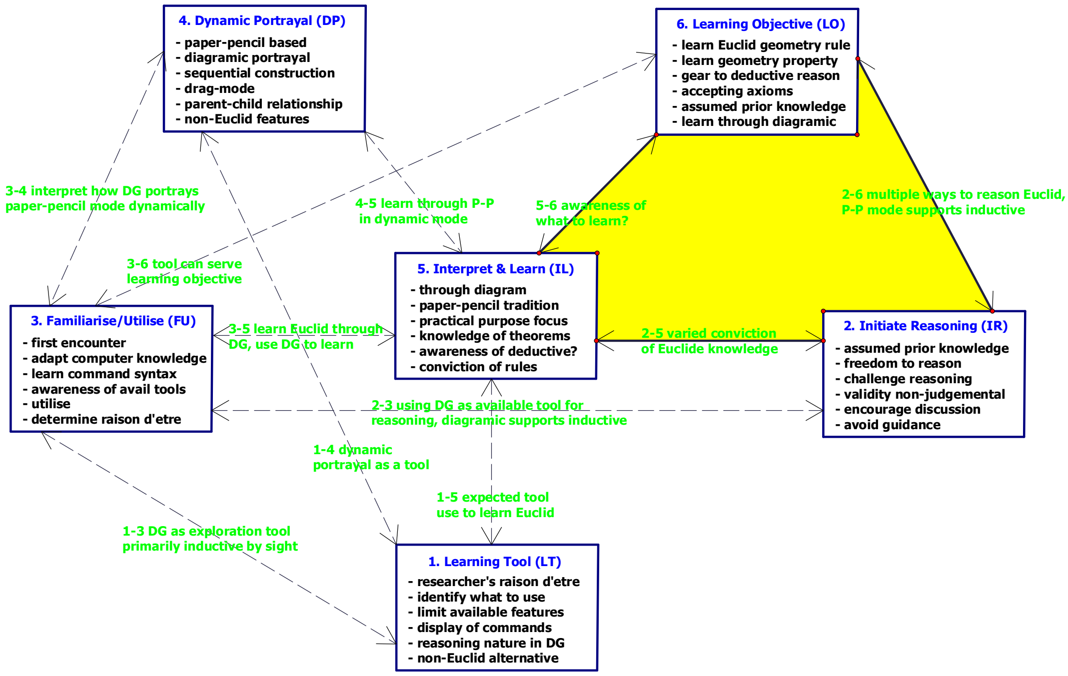


Figure 7.1 Inter-relationship 2-5-6

This chapter is divided into three sections: tasks designed for the pilot study phase, the analysis of the pilot study and the reflection and revision of the designed tasks for the main study

7.1 TASKS FOR THE PILOT STUDY

The activity for the pilot study is designed to formulate suitable geometric task for the main study. It adopts the same objective to investigate students' reasoning strategies to make sense of how DGS portrays Euclidean geometry and to perform the geometric construction and exploration tasks. Before the tasks are developed, the characteristics of the participant students in the pilot study need to be identified. Then the number of tasks would be designed accordingly to examine their suitability.

7.1.1 Participant Students

The pilot study is conducted with six Grade-8 students (14 years old). The rationale for choosing students at this age is that they should have learned basic geometry construction in the paper-and-pencil environment in Grades 6-7, and they have not yet been taught the more advanced geometric rules, such as circle theorem, which are included in the Grade 9 curriculum. This reflects the inter-relationship 2-3-5 in Figure 6.3, where assumed prior knowledge of geometry is expected to help the learners familiarise themselves with the Dynamic Geometry environment when they first encounter it. The six students are three boys and three girls and they are grouped into four different interview settings:

I) One boy interviewed individually

II) One girl interviewed individually

III) A pair of boys interviewed together

IV) A pair of girls interviewed together

The reason for the same-sex pairing is to encourage more dialogue between students, since students at this age are generally more comfortable working with the same-sex partner. When working in pairs, students are asked to share their software control by exchanging the mouse so that each of them can have the opportunity to interact with the software directly. All the students are of average-attainment level as assessed by their mathematics teacher. This is because the pilot study is set to evaluate the level of difficulty of each designed task, and average-attainment students should reflect the level of difficulty better than higher-attainment of lower-attainment students.

7.1.2 Development of the Tasks for the Pilot Study

Since the main criterion for the task design is that it should help the researcher pursue the answers to the research Sub-questions 1 and 2 during the main study phase. The central focus is, therefore, students' adoption of their prior geometric knowledge to reason the way DGS works and their reasoning strategies in the geometric construction and exploration tasks. The learning activity that helps the researcher gain these reasoning strategies is to let the students interact with the DGS by themselves and verbally express their reasoning processes. This kind of activity needs to be conducted through a series of geometric tasks which pose geometric challenges to the students, inviting them to experience the geometry knowledge by themselves. In order to design these tasks, the categorisation of task design in the dynamic geometry system discussed in Section 2.7 is used as a basis.

The research Sub-question 1 is set to investigate students' reactions to the DGS tool especially in how they interpret the software features, as well as their rationale for such interpretation. Of the seven approaches to geometric task design in the dynamic geometry system discussed in Section 2.7, the DGS Commands Familiarisation Task, aimed at letting the users familiarise themselves with the software's commands, appears to serve this purpose. However, besides letting the students try these DGS commands by themselves without instructions, the researcher would also play the role of a challenger, persistently asking them questions as to why they think each command works in such a way. The students are expected to adopt their prior geometrical knowledge as well as their reasoning strategies to explain the behaviour of the DGS commands they observe, which are the GSP commands in this case. These challenges add a reasoning dimension to the DGS Commands Familiarisation Task identified in Laborde's study (2001) which is more technologically-oriented. This type of task is referred to as Task 1 in this research.

Due to the limited time-frame, the learners are introduced only to some GSP commands. These commands include control commands and construction commands related to Euclidean geometry. Commands not directly related to Euclidean geometry, such as those in the Transform, Measure and Graph menus are excluded from this research. As a researcher, I do not inform the participant student of the name of each command nor give any hint of what these commands are for. I just identify the commands students should learn and let them try all these commands themselves. The list of commands to be introduced in to the students in this task presented in Section 5.7.

After the students learn to use these identified commands, they are invited to use GSP to draw any picture they like using as many commands as possible. This is to see how the students make use of each command, and how well they understand the way commands in GSP work. Students are also asked to verbally explain what they are trying to do at each step. This helps the researcher to understand the learner's view of DGS, especially its raison d'etre, and what they think DGS is actually for.

For the research Sub-question 2 which is set to examine students' reasoning strategies in the DGS environment, the approaches to task design to accommodate this query include:

- DGS Conditions Familiarisation Task which asks the users to construct a robust geometric figure via an awareness of the parent-and-child relationship in the DGS environment, and then justify the construction, providing the opportunity for reasoning.
- Heuristic Exploration Task Leading to the Reasoning Process which clearly demands the learners' reasoning strategy to explain the observed geometric property of given DGS constructions.
- Contradiction Investigation Task Leading to the Reasoning Process which adds an element of contradictory cases to the Heuristic Exploration Task to enhance the reasoning process.
- Uncertainty of Construction Task Leading to the Reasoning Process which adds an element of uncertainty of construction under constraints to enhance the reasoning process.

Nevertheless, as discussed in Section 2.7 the Contradiction Investigation Task Leading to the Reasoning Process and Uncertainty of Construction Task Leading to the Reasoning Process approaches to task design involve biased anticipation that students will follow a certain path to encounter such contradictions and uncertainties. In order to leave the choice of reasoning strategy entirely to the students, this research does not adopt these two approaches of task design but sticks to the DGS Conditions Familiarisation Task and Heuristic Exploration Task Leading to the Reasoning Process only.

The DGS Conditions Familiarisation Task approach, which usually focuses on the condition of parent-and-child relationship to construct a robust figure, can be adapted to become a geometric reasoning task by adding the challenge of geometric property to the otherwise technologically-oriented task. The fact that students need to understand the geometric property of the figure in order to construct it with the GSP tool encourages them to apply their prior geometrical knowledge as well as their experience with the tool. Students also need to reason in order to justify their choice of commands, construction strategies, as well as the final product, to ensure that it actually satisfies the task instruction. This part of the task is referred to as Task 2 in this research and involves the construction of a robust equilateral triangle, an isosceles, a rhombus and a parallelogram using only the commands available in the Tool Box and Construction menu for four different groups of participant students in the pilot study.

As discussed in Sub-section 2.4.6, the problem-solving activity can be used to encourage students' reasoning strategy, especially when they are challenged to verify their choice of strategy as well as their solution. The DGS Conditions Familiarisation Task can also be enhanced by adding more conditions to the robust geometric construction task in the GSP in order to provide more challenging scenarios to initiate a higher level of reasoning process from the students. This task demands more knowledge of geometric properties and a more complicated reasoning process to verify the construction to help the researcher to gain a better insight into the students' reasoning process under different circumstances. This task is set as a final challenge to the students and is referred to as Task 4 in this research. The geometric challenge in Task 4 includes the robust construction of a new circle tangent to a given circle at a given point, a circle circumscribing around a given triangle, a square inscribed in a given circle, and a new chord of equal length to the given chord of the same circle for four different groups of the participant students in the pilot study.

Another approach to task design to help the researcher gain an insight into the students' reasoning strategy in the DGS environment is the Heuristic Exploration Task Leading to the Reasoning Process where students are supposed to explore a geometric figure, discover a certain property and then reason to verify such discovered property. This approach to task design provides a situation where students may come across a surprising invariant property under the dynamic environment in the GSP when the figure is dragged, and then adopt reasons to explain why the figure behaves in such a way. Though the students are given the freedom to reason in their own way, the researcher plays the role of a reluctant believer, challenging them to provide better reasoning, especially through the deductive process of the observed phenomena. The students also have an opportunity to complete the partially-constructed figure to help them appreciate the order of construction affecting the parent-and-child relationship of the figure. This task is referred to as Task 3 in the research, and includes an exploration task of a triangle in a semi-circle, the property of the perpendicular bisector of a chord, the triangle midpoint theorem and the quadrilateral midpoint theorem for four different groups of participant students in the pilot study.

For the other two approaches to task design in the DGS environment discussed in Section 2.7, the objectives are rather pedagogical and do not actually conform to the purpose of this research, since I have no intention of teaching the students. The Geometric Concept Introduction Task aims to use the portrayal of geometric objects in the DGS environment to introduce new geometric concepts to the students. Though knowledge is supposed to be gained through the students' own analyses of the figures rather than from the teacher, the ultimate purpose of the task is to help students learn a new concept which is not the objective of this research. For the 'Black-box' Analysis Task, it appears to be an assessment tool to evaluate the students' understanding of the concepts taught, as well as the DGS features through the process of figuring out the steps in commands used to construct a given figure. Its purpose is also rather pedagogical and appears not to be related or useful to this research.

7.2 ASSESSMENTS OF THE DESIGNED TASKS

The tasks selected in the previous sections are trialled by the researcher with six 14-year-old grade-8 students: an individual boy, an individual girl, a pair of boys and a pair of girls. Each of these groups is given a different set of tasks, especially in Tasks 2-4, in order to evaluate their difficulty and suitability for the main study phase. These four groups of students are referred to by pseudonyms as follows:

Peter: an individual boy

Rob and Ryan: a pair of boys

Pauline: an individual girl

Ruth and Rachel: a pair of girls

The summary of the students and the tasks presented to them is given in Table 7.1.

Table 7.1 Tasks given to each group of students

	PETER	PAULINE	ROB+RYAN	RACHEL+RUTH
TASK 1	DG Exploration			
TASK 2: Construct	Equilateral Triangle (2a)	Isosceles (2b)	Rhombus (2c)	Parallelogram (2d)
TASK 3: Explore	A triangle in a semi-circle (3a)	Perpendicular bisector of a chord (3b)	Quadrilateral midpoint theorem (3c)	Triangle midpoint theorem (3d)
TASK 4: Problem	Tangent Circles (4a)	A circle circumscribes a triangle (4b)	A square inscribed in a circle (4c)	Equi-length chords (4d)

In order to let the participant students approach the tasks as intuitively as possible, the researcher emphasises to the students that the interview is not part of their academic assessment. They are free to perform the tasks in their own way without any pressure to provide the correct answers. They are encouraged to do whatever they think and try whatever they like with the tasks and they are encouraged to ask a question if any instruction is unclear to them. To lessen the interference effect, the researcher leaves the students to work on their own for a while during tasks

2-4. This is to let them contemplate the task on their own and feel free to experiment without the presence of the researcher. However, the researcher returns once they achieve a solution, or if time is running out, in order to ask for their reasoning.

For Peter and Pauline who work individually, the researcher asks them to perform the task by thinking aloud, i.e. to persistently express verbally what they are thinking and not to be shy. For Rob and Ryan, as well as Rachel and Ruth, the researcher asks them to share the mouse control and to discuss the task as much as possible in order to elicit their thinking and reasoning processes.

Fortunately, the researcher will have the opportunity to re-interview all participant students the day after the task-based interview for 15 minutes. This helps the researcher to clarify ambiguous or unclear points from the recorded task-based interview to better understand their reactions to the tasks.

The data gained from this pilot study phase is preliminary analysed in order to see how the designed tasks serve the purpose of investigating research sub-questions 1 and 2. The main coding used is the reasoning characteristics students adopt to perform each task to see if the tasks help elicit the students' reasoning process in any way. The students' overall performance is also assessed in order to evaluate the difficulty of each task. The different setting of interviewing individual students and a pair of students is then compared. The data analysis of the pilot study is, therefore, given by task rather than by reasoning type. This is to improve the researcher's evaluation of the design of each task. The analyses of students' performance of the four tasks are discussed in the following sub-section.

7.2.1 Students' Responses to Task 1

Task 1 aims to investigate the students' reasoning strategy to make sense of how commands in GSP work. Students are supposed to draw on their prior knowledge of Euclidean geometry to interpret each GSP command. From the data gained in the pilot study, students adopt various strategies in order to understand GSP commands.

For commands in the toolbox presented as graphic icons, most students realise the function of each command by trialling them for a couple of times before concluding what each command can do for them. For example, the Point tool is used to construct points and the Compass tool is used to construct a circle. This may be interpreted as a conclusion of inductive reasoning, where the common property of the multiple output of a particular command helps the students to generalise, despite the small amount of cases. The students may also draw on their experience with the common characteristics of other computer software commands where each command usually has a single unique function. A trial of only a few cases can, therefore, confirm the commands' operation.

Another instance is where some students realise the function of particular commands immediately after they first click the command on screen even before completing it. Rob and Ryan recognise the function of the Selection Arrow tool immediately after they see the rectangle frame generated by clicking and dragging the command across the screen. They realise that this command is used to select objects on the sketch even before they construct any object. They even use the term 'black cover' to name the particular function. When asked what they mean by 'black cover', Rob and Ryan explain that it is like doing a text selection in a word processing programme despite the fact that the selected object in GSP is highlighted in pink rather than in black. Rachel and Ruth also recognise the function of the Text tool immediately after they see a rectangle box and a blinking curser. They report that they have seen a command like this before in the Paint programme. These inferences by students to other software they have experience of may be regarded as abductive reasoning, based on similar appearances of the interactive command curser shared by GSP and other software. Such similarity leads the students to abduct that they should perform the same function, though there is no logical confirmation that similar-looking cursers would do the same task. This reaction reflects the inter-relationship 1-3-5 in Figure 7.1 where students' past experience of computer usage may play a part in how they make sense of GSP commands.

Though all participant students in the pilot study phase can generally realise the basic functions of commands in the toolbox, the way they describe the function of each command reflects their perception of software purpose. Ruth refers to points constructed by the Point tool as 'starting point', implying that more construction is needed in order to complete the process. Peter doubts that the Point tool is used to create a point on a line and not a free point, while Pauline interprets the Segment tool as a command to construct a radius for the circle she has just drawn with the Compass tool. These responses suggest that students believe these commands to be basic procedures for more meaningful construction and not very useful on their own. They show the students' process of reasoning in order to justify the presence of the commands in the GSP. This reaction reflects the inter-relationship 1-3-5, where setting GSP as a learning tool encourages the students to realise its purpose.

The layout of commands in the toolbox can also challenge students' imprecise identification of geometric entities. An obvious example is the Straight-edge tool which includes a Segment construction command, a Ray construction command and a Straight Line construction command (See Figure 7.2). The Thai mathematical term for 'Segment' is called 'ส่วนของเส้นตรง' which literally means 'a part of a straight line' but Thai students usually confuse it with the term 'straight line' since the Thai mathematical term for the 'straight line' is usually used to describe a segment in everyday life. When the participant students try the first Straight-edge tool command which is the Segment construction command, all of them call it 'a straight line'. But when the researcher invites them to try the other two commands: a Ray construction command and a Straight line construction command, they later realise that there are three different straight geometric objects and try to recall the correct terms for these commands. This situation demonstrates that GSP's organisation of commands influences the students' recollection of geometric knowledge. This shows the tension in the inter-relationship 3-5-6, where students interact with Euclidean geometry knowledge through the unique environment mediated by the DGS.

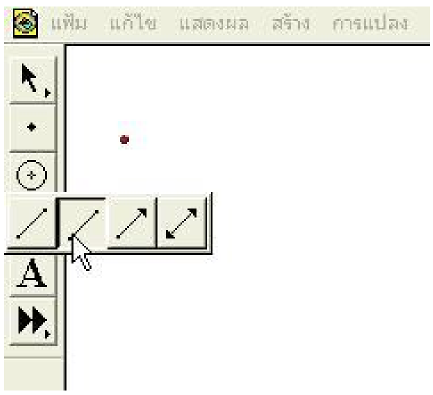


Figure 7.2 Three commands in Straight-edge tool

After the students become familiar with the basic commands in the toolbox and can use them to construct a simple geometric figure, the researcher encourages them to try to delete some of the objects they construct. All groups of students can deduce from their past experience with other computer software that the Delete or Backspace button on the keyboard can be used to delete the object. They also realise that the object to be deleted needs to be selected by the Selection Arrow tool before the command can be executed. When asked for a reason, students explain that they had to specify the object to be deleted first or else the software would not know what to delete. This explanation illustrates an insight where the students start to imagine the process of GSP. It reflects the inter-relationship 1-4-5 where students learn GSP's algorithm in order to manipulate Euclidean geometry objects.

For most commands in the Edit menu, students can immediately interpret the function from the title of each command. Some of them also infer the knowledge gained from other computer environments to elaborate that the Copy and Paste commands should be used together to construct a duplicate of an original figure. This confirms the inter-relationship 1-3-5, where prior knowledge in the computer environment helps them to make sense of how the DGS works. However, there is some difficulty when students are asked to elaborate on the distinction between the Cut command and the Clear command; most students do not recognise the difference since both appear to eradicate the object from the screen (the fact that the Cut command moves the object from screen to the clipboard is invisible to the students). Peter first believes that the Clear command cannot be undone, while the Cut command can. He realises later that this is not the case. Rob, Ryan, Rachel and Ruth, on the other hand, simply conclude that these two commands execute the same function. Only Pauline distinguishes the Clear command from the Cut command by explaining that the Cut command can be pasted later while Clear cannot. This behaviour shows the students' different reactions to the redundancy of the software commands, especially when commands with similar actions appear in the same menu.

Most basic commands in the Display menu are also intuitive to students, especially the Line Width and Colour commands, and can understand and make use of these functions immediately. The Hide command provides a good example where a verbal title can help students realise the command functions. Students can deduce from their understanding of the term Hide to distinguish it from the Cut or Clear commands even though these all make the object disappear from the screen. They can also pair the Hide with Show All Hidden command as opposite to each other.

Commands in the Construct menu cause more difficulty in understanding for the students especially when they are faced with the condition that a correct set of objects needs to be selected before the command becomes enabled. Sometimes students' inaccurate knowledge of geometric terms prevents them from realising the appropriate input for some commands. For example, Rob and Ryan mistake the Midpoint command as one to construct the centre of the circle (the terms 'midpoint' and 'centre of a circle' sound very similar in Thai). They keep trying to construct a midpoint for a circle again and again without success. Pauline does not recall the definition of an 'intersection' so she ends up adopting a trial and error strategy by randomly selecting various combinations of objects until she gets the command enabled and realises that is it used to construct intersection point(s) of intersecting objects. Nevertheless, these students understand that the commands are disabled because inappropriate input was selected, though they did not know what was needed in the first place. After several attempts, all students manage to find the appropriate input for each command.

Peter, Rachel and Ruth also adopt deductive reasoning based on known properties of geometric shapes to explain the GSP commands' behaviour. When they find that the Midpoint command cannot be applied to a ray or a straight line and remains disabled. They explain that the software cannot calculate the midpoint of these objects since their lengths are infinite and hence the midpoint cannot be positioned. The reaction also reflects the inter-relationship 1-4-5 where the students' knowledge of Euclidean geometry helps them to make sense of how GSP operates.

The conflict between the students' expectation of how a GSP command should work and how it actually operates can be found in their experience with Parallel Line commands. All students in the pilot study phase expect the Parallel Line command to adjust two non-parallel segments to become parallel with each other. They realise later that the command does not function that way and adopt a trial and error approach before finding that a point and a straight object is needed for the Parallel Line command. This example shows that students have an idea of what a Parallel Line command should give, i.e. a pair of parallel straight objects, but they have a different idea of how the software should execute a Parallel Line command from the designer. The students' expectation implies that they appreciate the dynamic feature in GSP as a facilitating tool to help them adjust the figure in the way they want, i.e. turning non-parallel segments into parallel segments. This contrasts with the GSP designer's approach where the Parallel Line command is used to construct a new parallel line to the existing straight object based on the Euclidean geometry concept. The original straight object can then be moved and the constructed parallel line dynamically responds to the original object's movement to keep the parallel property. The students' notion of the dynamic feature as a tool to generate geometric property, and the designer's notion of the defined property which retains in dynamic environment, exemplifies the conflicting ideas between the user and the designer, leading to unmatched interpretations of the commands.

Based on these reactions to Task 1, it can be seen that allowing the students to learn how each command in GSP works without any guidance challenges them to adopt various reasoning strategies as well as prior knowledge, both in Euclidean geometry and computer software usage, in order to interpret the GSP environment. This interpretation is independent of the designer's intention and the difference stems from preference or personal judgement of how DGS should portray dynamic Euclidean geometry. The basic approach of Task 1, where students are encouraged to try and learn to use each command in GSP by themselves, provides a helpful situation for examining their reasoning strategy, as well as their background of Euclidean geometry knowledge.

7.2.2 Students' Responses to Task 2

Task 2 involves the students' construction of a basic geometric figure with a condition that the constructed figure retains its property or cannot be messed-up when dragged. From the four designed tasks of constructing an equilateral triangle, an isosceles, a rhombus and a parallelogram as summarised in Table 7.1, the only group of students to successfully construct the figure that cannot be messed-up is Rachel and Ruth who are supposed to construct a parallelogram. Note that this group of students is the only group that the researcher allows to use any command they learn in the GSP, while the other three are restricted to Compass and Straight-edge tool commands.

When asked what a parallelogram is, Rachel and Ruth explain that it is a shape that has two parallel pairs of sides and the parallel sides are equal to each other which satisfies the basic property of a parallelogram. In order to construct this shape in the GSP environment, Rachel and Ruth use the Straight-edge tool to construct two segments forming an acute angle as two adjacent parallelogram's sides where the first segment lies horizontally. Then they use the Parallel Line command to construct a new straight line parallel to the horizontal segment through the top end point of the second segment. At this point, Ruth finds that the infinite length of the parallel line is problematic and discusses with Rachel the way to cut the extended length. Note that the researcher has informed them in advance that a line of any length can be used for the construction. Rachel seems not to worry about the extended length of the parallel line and keeps concentrating on the next step to construct the fourth side. Rachel then uses the Segment tool in the Straight-edge command to construct the fourth side parallel to the second segment by eye (see Figure 7.3).

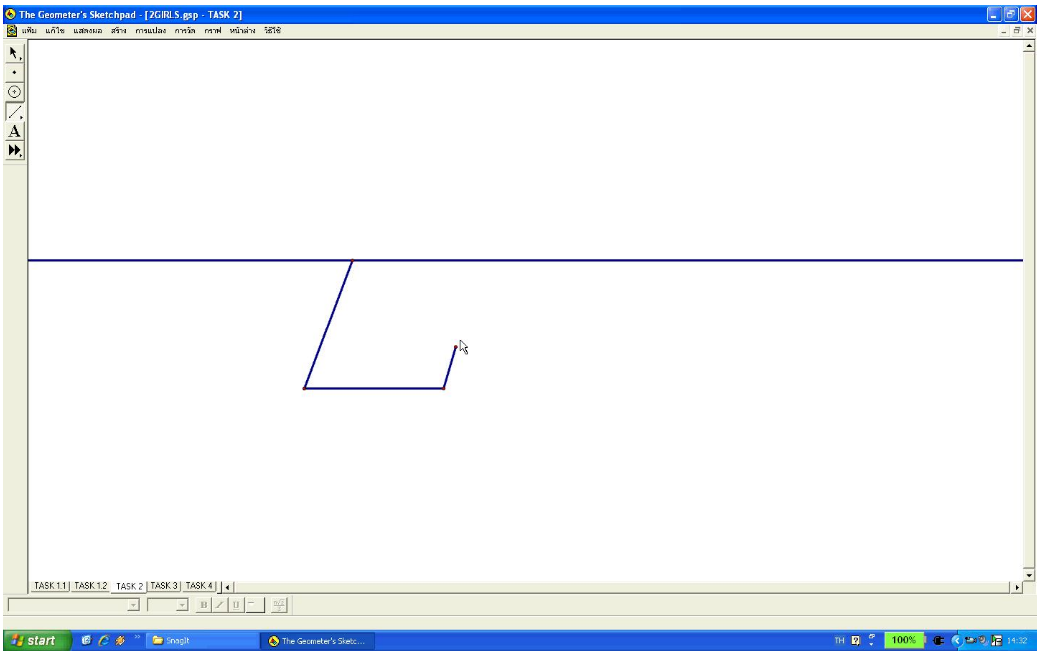


Figure 7.3 Rachel and Ruth's construction of a parallelogram

Below is the dialogue between Rachel and Ruth at this point.

Ruth: How do you know that they are parallel to each other?

Rachel: Mmm.

Ruth: Let's try to construct a parallel line.

Rachel: How?

Ruth: Select this side [the left side in Figure 7.4] and this point [the right end-point of the bottom side in Figure 7.4] and use the Parallel Line command.

Rachel: (laughs) is that it?

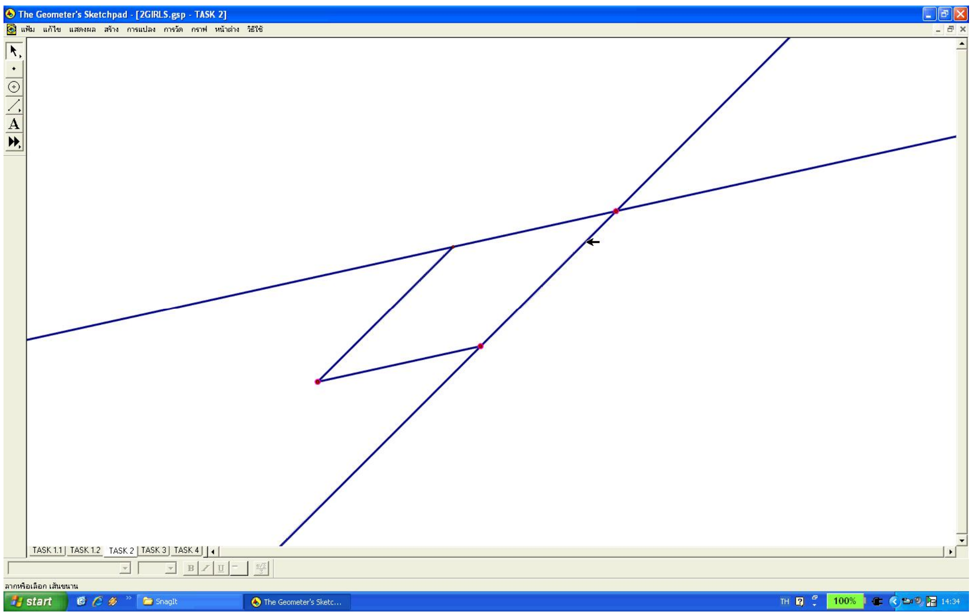


Figure 7.4 Rachel and Ruth complete the parallelogram construction

Once Rachel and Ruth complete the construction, Ruth uses the Arrow Tool to drag the left side of the quadrilateral randomly and they both notice that the two pairs of sides remain parallel to each other. Ruth concludes that the construction should be a parallelogram though she says that she does not know how to make the infinite sides shorter. The researcher then reassures them that an infinite line can be used as a part of the construction as long as part of the construction is a robust parallelogram. Ruth's uncomfortable reaction to the infinite parallel lines may stem from her experience in the paper-and-pencil tradition where a segment is commonly used to construct a geometric shape. An infinite parallel line as a part of the constructed figure in the GSP environment is therefore something alien to her even though it better reflects the definition of a geometric straight line.

When asked to verify why their construction is a parallelogram, Rachel and Ruth deductively explain that the figure is constructed with the Parallel Line command applied to the two adjacent sides giving two pairs of parallel sides which is the property of a parallelogram. However, when the researcher asks them to further verify that the parallel sides are equal in length, the girls simply reason that two pairs of intersecting parallel lines would cut at the same length without referring to other geometric property. After that, the researcher encourages them to drag each vertex to see the figure's behaviour. The responses from the girls are as follows.

Researcher: What happens when you drag this lower right vertex?

Rachel: The opposite vertex also moves away from the vertex I move.

Researcher: Why it is so?

Rachel: Maybe because they are opposite vertices.

Ruth: Let's try this top left vertex . . . The opposite vertex stays still this time, just the lines that move.

Rachel: Maybe because the lines are all connected.

Researcher: How about the top right vertex? What happens when you drag it?

Ruth: The whole figure moves!

Researcher: Why it is so?

Rachel: Maybe because these two parallel lines intersect at that point.

Ruth: Or maybe because it was the last point we constructed.

This response shows that the students see dependent connections between entities of the figures and realise that the order of construction can influence the constructed figure's behaviour. Rachel and Ruth, therefore, perceive the parent-and-child relationship in GSP as a temporal connection between objects constructed in order.

Peter's solution to construct an equilateral triangle with the Compass tool and the Straight-edge tool gives an interesting result. He succeeds in the geometric part of the construction but fails to respect the robust figure condition that it should not be messed-up under dragging. After clearly defining that an equilateral is a triangle with three equal sides, Peter constructs an equilateral triangle by using the Compass tool to construct three circles of equal size with their centres lying on the circumference of the other two circles. He then uses the Straight-edge tool to construct three segments connecting the centre of the three circles forming a triangle. When asked why he thinks the triangle is an equilateral triangle, Peter deductively reasons that the three sides are radii of three circles of equal size, so the three sides of the triangle are equal in length as shown in Figure 7.5.

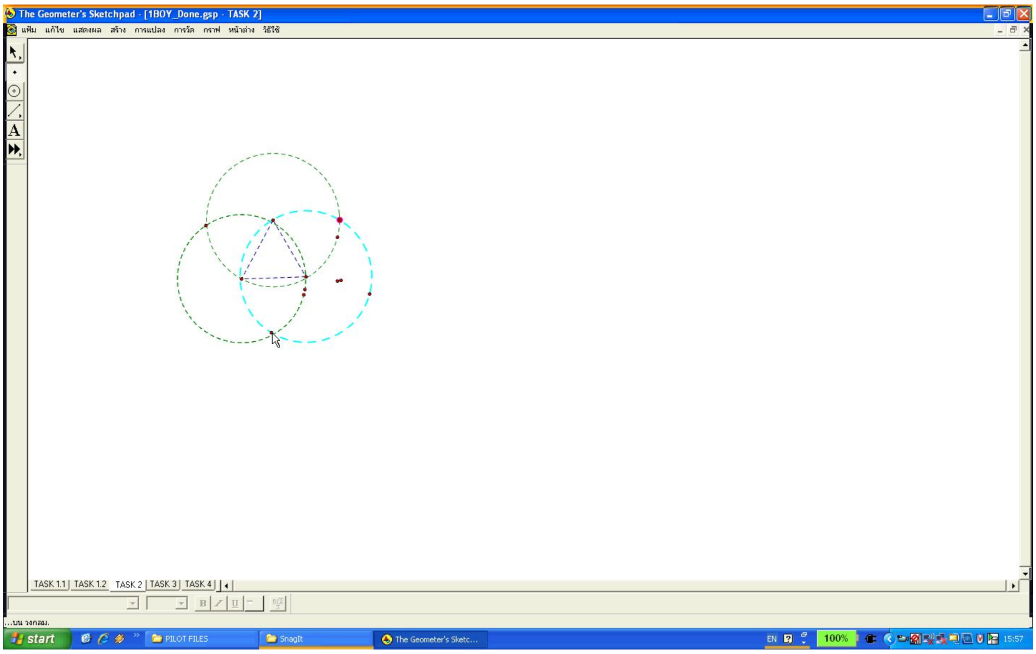


Figure 7.5 Peter's construction of an equilateral triangle

Nevertheless, when the researcher asks him to drag the vertices of the constructed triangle, the circles move apart and the equilateral triangle deforms. This is because when he constructs the three circles, he does not join the new circle’s centre with the point on the circumference of the existing circle. Peter tries to fix this by using the Point tool to create points at the circles’ intersections expecting the points to link the circles together but he realises later that the strategy is not fruitful.

Peter's response to this task illustrates the independence of the robust figure condition in the DG environment from geometrical reasoning since he actually accomplishes the geometric strategy and verification but fails to satisfy the dynamic condition. It distinguishes the software rules set by DG in order to construct a robust figure from the Euclidean geometrical rules. Though Peter's reaction shows that he manages to understand the geometric rule despite the failure to construct a dynamically robust figure, other students are supposed to encounter these software rules and geometric rules in the GSP environment at the same time and this may be confusing to them.

Peter's utilisation scheme of the Point tool as a tool to connect the figure together also shows his personal interpretation of the point's function in the DG environment. The visible points at the circles' intersections are not just points indicating the intersecting positions but they also tie the objects together. Points in the GSP environment are intentionally much more noticeable than points in the paper-and-pencil environment. This may be the reason that leads Peter to think that they should have a significant function other than showing the position of the intersections. Peter's anticipation of the intersecting points' behaviour resembles the students' reaction in Jones' studies (1996, 1998), where students think that the intersection point 'glues' the figure together.

In order to examine Peter's reaction to the parent-and-child relationship, the researcher instructs him to draw new circles connected to each other and asks him to construct an equilateral triangle again. When he completes the new construction as shown in Figure 7.6, the researcher invites him to drag each of the triangle's vertices. Peter reports and explains the triangle's behaviour as follows.

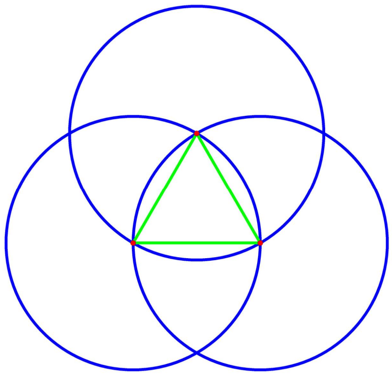


Figure 7.6 Peter's robust equilateral triangle

Researcher: What happens when you drag each of the vertices?

Peter: The triangle can be enlarged when the base vertices are dragged but it won't change the size when the top vertex is dragged.

Researcher: Why is it so?

Peter: Maybe because when we construct the top circle we join the circumference point with the circle on the left only. So when we move the top vertex which is the top circle's centre, the software cannot estimate the length of the triangle's right side. The whole figure then moves without changing the size.

This response illustrates Peter's perception of how the way the figure is constructed affects the figure's overall behaviour. He shows an insight into determining how GSP works and claims that the figure's different behaviour when the top vertex is dragged is as a result of insufficient parameters for GSP to calculate the position of the right side of the triangle (while in fact it is because the top vertex is the child or an intersection of the parent two circles). This reasoning shows how Peter tries to understand the software's algorithm by determining what is needed in order to display the figure correctly, especially how the flexibility of the drag-mode is limited by the software's capability.

The other two groups of students fail to construct a robust isosceles and a robust rhombus with the Straight-edge tool and the Compass tool in the GSP. These two commands are given as a guide to the students to make the task easier. However, both groups of students ask to use other commands: the Perpendicular Line command and the Parallel Line command, to construct the figure. The researcher decides to permit this in order to investigate their strategies. Pauline is instructed to construct a robust isosceles. She uses the Compass tool and the Straight-edge tool to construct a circle and its horizontal diameter. Then she asks to use the Perpendicular Line command to construct a perpendicular line to the circle's diameter through the centre. She then uses the Segment tools to complete the triangle as shown in Figure 7.7.

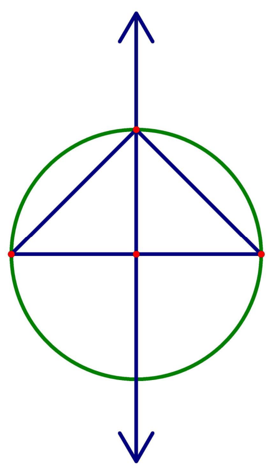


Figure 7.7 Pauline's construction of an isosceles

Similar to Peter, Pauline fails to construct a robust isosceles. However, she manages to deductively verify her construction by referring to the symmetric property of the constructed figure. Here is Pauline's verification of her isosceles construction.

Researcher: Why do you think it is an isosceles?

Pauline: The sides are equal.

Researcher: Which sides are equal?

Pauline: This side (points to the left side) and this side (points to the right side).

Researcher: Why are they equal?

Pauline: This perpendicular line is the circle's other diameter and it precisely splits the circle and this (horizontal) diameter into two halves, so the segments connecting the intersection of these perpendicular diameters and the circle should be equal to each other.

It should be noted that Pauline's construction is limited to a right isosceles which may not be flexible enough to be accepted as a common isosceles in the dynamic environment. Her approach is therefore to construct 'an isosceles' without the dynamic concern that it should be adjustable to any other isosceles shape. Nevertheless, her construction satisfies the isosceles' geometric property though she cannot manage to construct a robust and generalised isosceles in the DG environment.

Rob and Ryan are also unsuccessful in constructing a rhombus using only the Compass tool and the Straight-edge tool. They ask to use the Parallel Line command which the researcher also permits. Since the task requires them to use the Compass tool, Rob and Ryan try hard to fit the rhombus into a circle, especially a rhombus where two sides lie horizontally, which is commonly seen in the paper-and-pencil environment (see Figure 7.8). Rob and Ryan's adherence to this particular orientation of a rhombus prevents them from constructing the shape from two joining circles sharing a radius. This reaction shows how students' experience in the paper-and-pencil environment influences their strategy in the DG environment. It also shows how the guiding instruction to use the Compass tool command can mislead students especially when they have a different strategy in their mind, i.e. constructing a rhombus with the Parallel Line command in this case. Due to the unsuccessful attempt to construct a robust rhombus in the circle as intended, the researcher has no opportunity to ask for Rob and Ryan's verification of the construction.

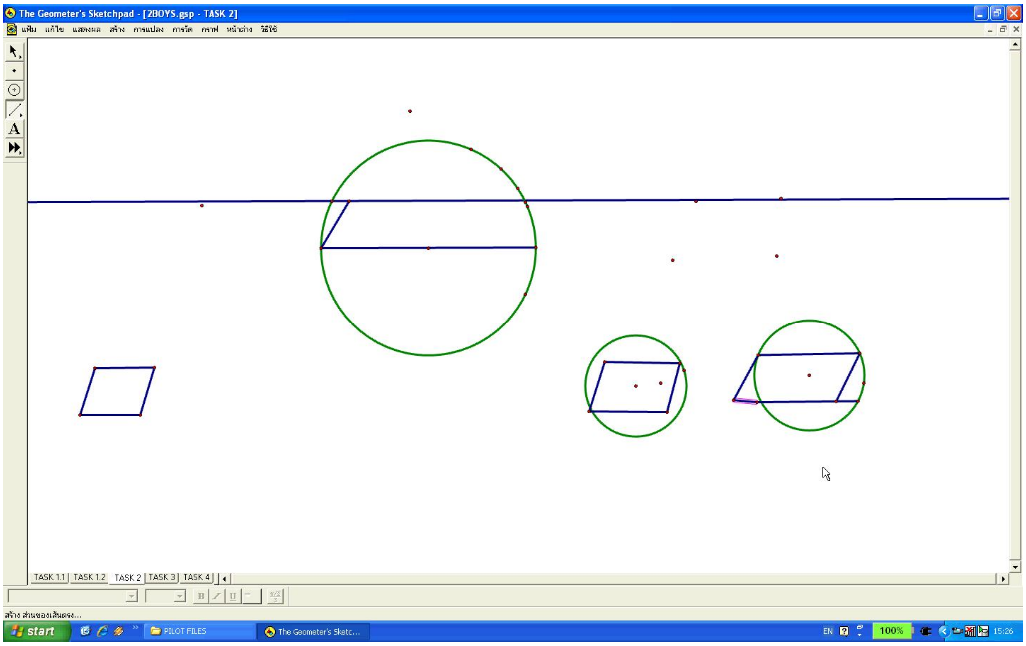


Figure 7.8 Rob and Ryan's strategies to construct a robust rhombus

The geometric construction task with the GSP in Task 2 helps the researcher to examine the students' knowledge of basic geometric shapes and their properties, and observes how they utilise the commands they learn in the GSP to construct a robust figure. Though the condition that the constructed figure should be robust concerns only the dynamic property of the figure in the DG environment and may be independent from the geometric verification process as shown in Peter's case. The fact that students have to encounter these two domains of property in the DG environment at the same time presents the circumstance that this research aims to investigate. The condition that the constructed figure must be robust should, therefore, remain in this task. Nevertheless, the condition of using particular commands tends to mislead students rather than helping them as anticipated by the researcher. It may be more appropriate to let the students use any command they like in order to investigate their spontaneous response. The construction verification process also helps the students to affirm that the geometric property and the commands they choose in GSP satisfy the given instruction. It also helps the researcher to elicit the students' reasoning strategy to verify their construction. The condition to construct a robust figure also allows the researcher to investigate the students' response to the parent-and-child relationship, provided they can construct the figure successfully. The geometric construction and verification tasks in Task 2,are therefore useful in studying students' reactions to Euclidean geometry portrayed in the GSP environment.

7.2.3 Students' Responses to Task 3

In Task 3, students are supposed to explore given geometric situations in the GSP environment. They are instructed to perform a simple additional construction to the pre-constructed figure, to explore the completed figure and are expected to discover certain geometric properties. They are then encouraged to give reason to the geometric property they discover from the observation. The geometric situations selected for the pilot study of this research include the triangle in a semi-circle, the perpendicular bisector of a circle's chord, the quadrilateral midpoint theorem and the triangle midpoint theorem. Students' responses to this task are divided into two distinct phases: the Exploration Phase and the Justification Phase.

Exploration Phase

After students perform the additional construction from the basic GSP commands which none of them find to be a problem, they are asked to explore the complete figure with the drag feature. During the exploration phase, the drag-modes that students use most are wandering dragging and bounded dragging. Their attention generally focuses on the relative changes of the figure's components when a certain point or part of the figure is dragged. For example, Pauline who performs the perpendicular bisector of the circle's chord task, reports that "when point R is moved along the circle's diameter (bounded dragging), the length of segment increases or decreases and will have the maximum length when point R comes closest to point O (see Figure 7.9).

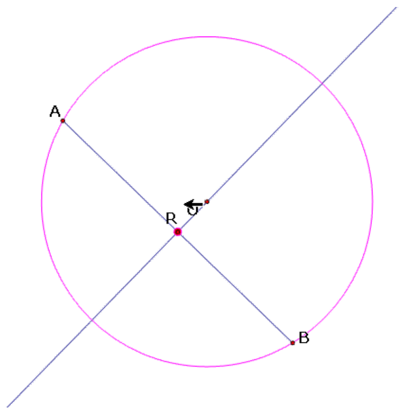


Figure 7.9 Pauline's bounded dragging

Peter who performs a triangle in a semi-circle task reports that the triangle can be moved by dragging point C along the circumference. However, moving point A would also move point B in the reverse direction but moving point B would just translate the whole figure (see Figure 7.10).

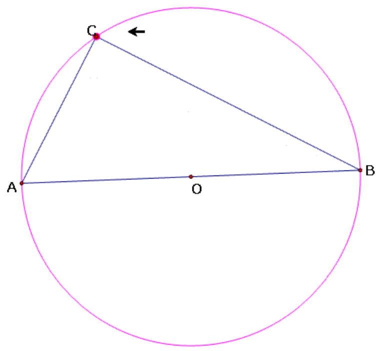


Figure 7.10 Peter's observation of Task 3a

When asked why the figure behaves in such a way when point A or B is dragged, Peter conjectures that it is because point B is constructed after point A, showing his appreciation of the parent-and-child relationship in the DG environment. Peter does not observe that triangle ACB is always a right triangle. When the researcher guides him to observe angle ACB, he responds that angle ACB would be greatest when point C is perpendicular to the diameter AB at point O and the angle is 90 degrees which is the biggest possible in a triangle (see Figure 7.11). However, he realises later by himself after bounded dragging of point C back and forth along the top arc between points A and B, that angle ACB is always 90 degrees. The drag-mode in DG helps Peter to examine the geometric situation thoroughly and more carefully.

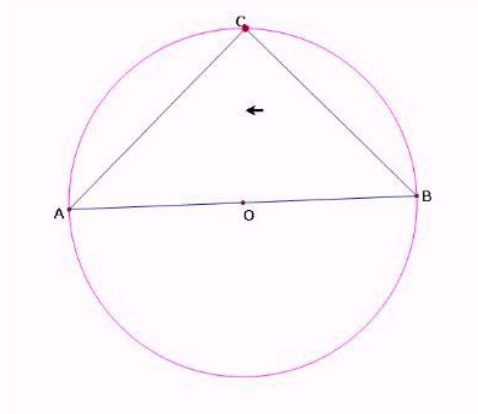


Figure 7.11 Peter's original claim of Task 3a

Rob and Ryan also use wandering dragging and guided dragging to explore the quadrilateral midpoint theorem. They drag each point around the screen randomly and claim that the size of the quadrilateral can be adjusted by dragging. They also claim that the outer quadrilateral can be adjusted into different shapes and use guided dragging to adjust the figure into a rectangle and a trapezoid. However, they fail to observe that the inner quadrilateral is always a parallelogram. The researcher has to encourage them to observe the inner quadrilateral's property when each point is dragged, but Rob and Ryan respond that the inner quadrilateral can either be a rhombus or a rectangle when the outer quadrilateral is a rectangle or a trapezoid respectively as shown in Figure 7.12.

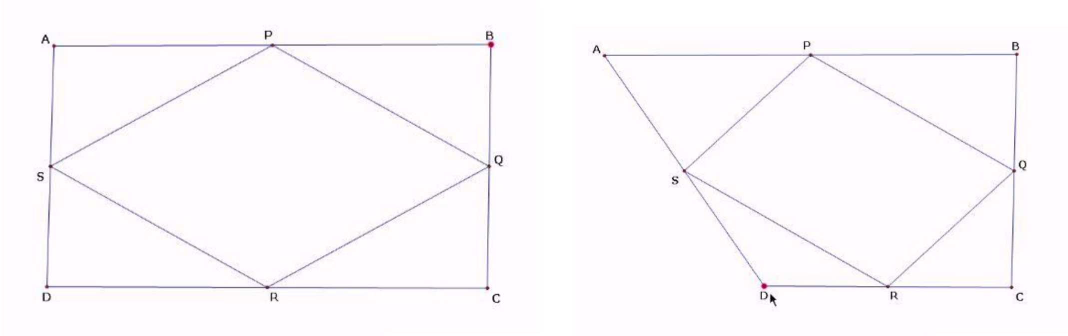


Figure 7.12 Rob and Ryan's imprecise identifications of the inner-quadrilateral

The pair fails to generalise that the inner quadrilateral is actually a parallelogram, possibly due to their exclusive appreciation of the geometric shapes' definition that a rhombus or a rectangle is not a parallelogram. Note that they roughly identify the inner quadrilateral by eye without actually referring to its property. They claim that the inner quadrilateral is a rectangle when the outer quadrilateral is a trapezoid even though it is obvious that the angles are not right angles. Rob and Ryan's use of guided dragging to adjust the figure into different shapes helps them explore the property of the construction, though they cannot generalise the geometric property of the quadrilateral's midpoint theorem since they clearly distinguish a rhombus and a rectangle as different shapes, based on their prior geometry knowledge. Nevertheless, when the researcher guides them to observe the relationships between sides SP-QR and PQ-RS, Rob and Ryan notice that these pairs of sides are always parallel though they never explicitly identify the inner quadrilateral PQRS as a parallelogram.

On the other hand, Rachel and Ruth recognise immediately after completing the additional construction that the two triangles are similar. They abductively refer to the lesson on similar geometry they had just learned in the class using the same shape, and also claim that they are similar because there are parallel lines and a common angle. However, when the researcher asks them how they know that the segments are similar, Rachel and Ruth use wandering dragging to further explore the figure by eye and claim that they still look parallel.

The drag-mode in the GSP, therefore, helps students observe the variant and invariant property of the figure when a point or part of the figure is dragged, though they generally need some guidance from the researcher in order to find out the intended rule or property. The flexibility of the drag-mode also helps them to adjust the figure in the way they want in order to observe the result. It also allows students to see the gradual changes of the figure helping them to closely examine the situation. The students can therefore, generally discover the intended geometric rule in this designed task.

Justification Phase

After the students discover the intended geometric properties, they are challenged to explain the reason why the constructed figure behaves in such a way. Students are supposed to use the GSP's commands and drag-mode to help them reason. From the students' performance of the task in this phase, a range of reasoning strategies is adopted in order to verify the discovered properties. They are discussed by different types of reasoning as follows.

i) Inductive Reasoning

The basic type of reasoning commonly found among these groups of students in the justification phase is inductive reasoning by using the GSP's drag-mode to visually illustrate the property observed. This type of reasoning is an overarching strategy that Rachel and Ruth use to justify the triangle's midpoint theorem.

When they recognise that the two triangles in the diagram are similar triangles with a common angle ABC and parallel sides AC and PQ, the researcher asks them to verify that AC and PQ are actually parallel to each other. Rachel first simply states that they are parallel because she 'sees' them parallel. She then further justifies this parallel property by using the guided dragging to adjust the triangles into an unusual shape, i.e. dragging side AC up beyond point B which she calls a 'reversed figure', showing that AC is still parallel to PQ as can be seen in Figure 7.13.

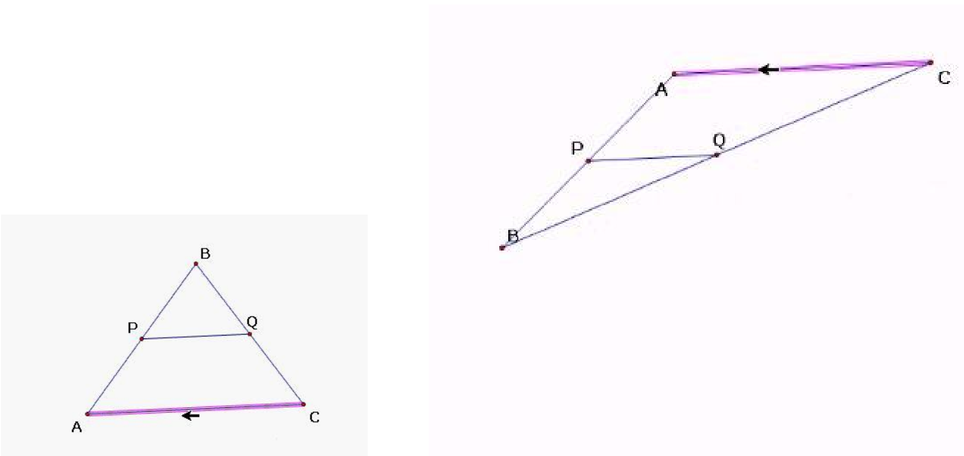


Figure 7.13 Rachel drags side AC upward beyond point B

Rachel's strategy can be deemed to be a Crucial Experiment, or using a particular case to testify that the conjecture is still valid as defined by Balacheff (1988). She illustrates that the parallel property is still true even if the triangle is adjusted to a different arrangement, e.g. in a ‘reversed figure’ as she refers to it. Such a Crucial Experiment with a special case can be deemed to be an inductive reasoning stage of the overall abductive reasoning strategy to pose a hypothesis and then verify it as discussed in Sub-section 2.4.4 in the Literature Review chapter.

Pauline’s strategy to verify the perpendicular bisector property of a circle’s chord involves a special kind of inductive reasoning called ‘transformational reasoning’ where a series of empirical evidence is considered dynamically rather than in a collection of separate cases (Simon, 1996). When asked to verify that the perpendicular bisector of a circle’s chord always passes through its centre, Pauline uses the drag-mode in the GSP to drag the chord’s endpoint B down along the circumference until the chord appears to be the circle’s diameter where Point P overlaps the centre O as shown in Figure 7.14.

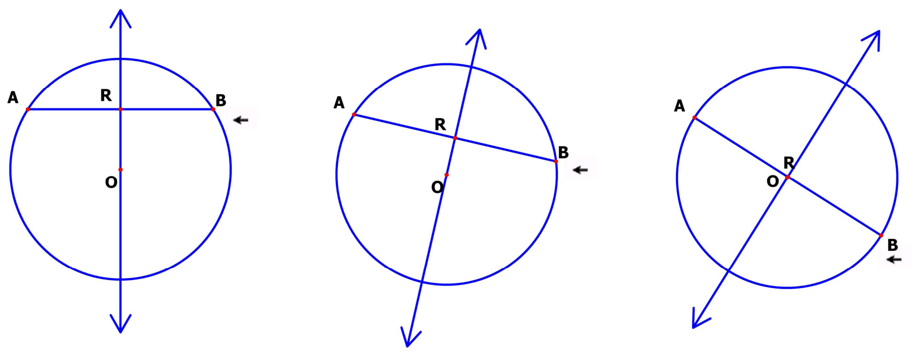


Figure 7.14 Pauline's transformational reasoning strategy

Here is Pauline’s explanation of the case:

Researcher: Why do you think this perpendicular line always passes point O?

Pauline: If we keep enlarging the chord AB, point R will still be midpoint. Until the chord is biggest it will become a diameter and the midpoint R would become the centre O like this.

Though Pauline's response does not refer to the fact that the straight line is perpendicular to the chord AB, she manages to show the gradual change of the chord AB when it is enlarged and to point out the fact that point R always lies at the centre of chord. She uses the dynamic feature in the GSP to visually illustrate that chord AB can be adjusted until it becomes the circle's diameter, and point R shares the same position with the circle's centre O. The transformational reasoning is therefore used by Pauline to show the relationship between the circle's chord and its diameter.

ii) Deductive Reasoning

Several justification strategies by the students can be categorised as deductive reasoning in Task 3. The most basic form of deductive reasoning found in the pilot study is the direct deduction from geometric definition or tautology. When asked to verify why a triangle in a semi-circle is a right triangle, Peter simply refers to the property of the right triangle: 'because the angle is always 90 degrees'. Though this tautological response does not give any logical inference, it still shows the student's geometric background knowledge and may broaden strategies for further reasoning, especially when an algebraic operation is performed with the fact that the angle is always 90 degrees. For example, the student may relate this fact to another known fact, that the sum of the internal angles of any triangle is always 180 degrees, to further explain the case.

However, when challenged by the researcher to verify why the angle is always a right triangle, Peter manages to adopt more sophisticate deductive reasoning based on his prior knowledge of geometric properties. In order to verify the property, Peter first arranges the triangle into an isosceles as shown in Figure 7.15, and contemplates that OC should be another circle's radius and the researcher encourages him to use the GSP construction command to draw OC.

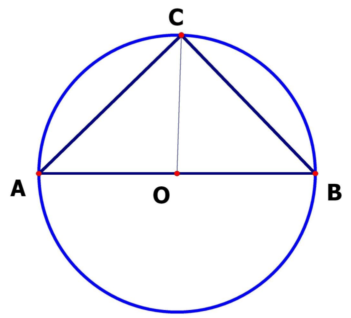


Figure 7.15 Peter starts reasoning from an isosceles

From Figure 7.15, Peter claims that the triangle ACB is actually half a square. The researcher therefore advises him to construct the other half of the square with the GSP commands as shown in Figure 7.16, and asks him to verify why he thinks it is a square.

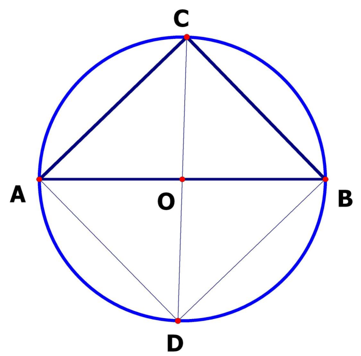


Figure 7.16 Peter reasons by completing the square

From Figure 7.16, Peter then deductively refers to the property of the diagonals of the square that they are perpendicular and bisect each other. Since OA, OB, OC and OD are all radii of the same circle and CD is the extension of radius OC intersecting the circle's circumference, AB and CD are therefore equal and bisect each other making ACBD a square. Note that Peter omits another essential diagonal property; that they should also lie perpendicularly to each other for the square case. The researcher then further challenges Peter to verify the case where C is moved to a different position, where Peter responds that ACBD is still a rectangle, and the property that the diagonals of the a rectangle are equal and bisect each other remains as shown in Figure 7.17.

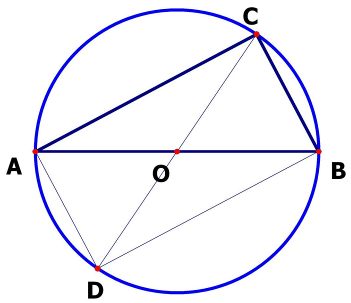


Figure 7.17 Peter's verification of a rectangle case

Peter's strategy to justify that ACB is always a right angle makes a clear deduction from the property of a rectangle's diagonals. It is interesting that Peter starts his strategy from a symmetric isosceles shape where vertical and horizontal symmetric properties of the figure may help him easily envision the square through reflection. He then adopts that same strategy to explain the more generalised case where ACB is no longer an isosceles, by verifying that ACB is still a right angle and ACBD is a rectangle from its diagonals' property.

iii) Abductive Reasoning

Apart from using basic abductive reasoning to pose a hypothesis that the observed property is always true, some students also adopt a strategic form of abductive reasoning to look for possible relevant information to help them justify the hypothesis. While working on Task 3c to explore the quadrilateral midpoint theorem, though Rob and Ryan do not clearly state that the internal quadrilateral is always a parallelogram, they manage to point out that the opposite sides of the internal quadrilateral are equal and always parallel to each other as shown in Figure 7.18.

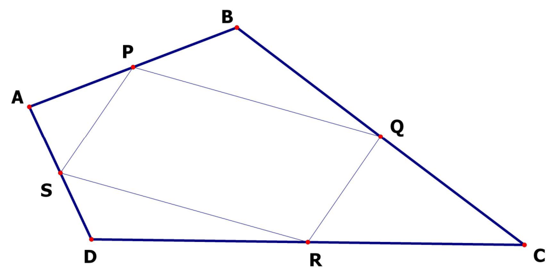


Figure 7.18 Rob and Ryan's abduction to the midpoint constructions

The researcher then asks them to verify this conjecture and they adopt abductive reasoning by referring to the fact that P, Q, R and S are all midpoints of ABCD quadrilateral’s sides and this should make PS equal to QR and PQ equal to SR, though they fail to find more information to fully support this claim. Though their identification of the fact that P, Q, R and S are midpoints is a relevant factor for the observed properties may be directly inferred from the construction process, Rob and Ryan’s focus on these property points still implies that the figure would behave differently had P, Q, R and S not been midpoints. This abduction has the potential to lead them to further investigation according to this specific property, though they do not manage to complete the justification in this task.

From the students' responses to Task 3, both in the Exploration and Justification phases in this pilot study, it is evident that the participant students at this age can manage to discover and then adopt various reasoning strategies to verify the claim, with challenges and some guidance when needed from the researcher. The design tasks and the selected approach to keep challenging students for better reasoning can therefore help to elicit students' reasoning strategies which can be further analysed in how these strategies are related to the GSP environment.

7.2.4 Students' Responses to Task 4

Task 4 aims to examine the students' strategies and their reasoning in the geometric problem solving task in the GSP environment. Students are presented with geometric construction tasks with constraints and they are supposed to give a reason why their solution conforms to the given instruction. This task is also expected to reflect students' utilisation schemes of the GSP in order to solve a problem, which is portrayed as inter-relationship 3-5-6 in the analysis model.

An interesting strategy that students adopt in the problem-solving task is using abductive reasoning to devise a likely solution, and then trial such a solution to see whether it gives the desired construction. When asked to construct the smallest circle covering a triangle, Pauline first estimates the position of the circle's centre by using the Compass tool to construct a circle, with the centre somewhere around the triangle's centroid. She then undoes the construction and uses the Midpoint construction command to construct midpoints for all three sides of the triangle. After that

Pauline uses the Segment tool to construct the triangle's three medians and uses the medians' common intersection as a new circle's centre. However, Pauline realises from this strategy that the circle with the triangle's median intersection point as a centre is not the smallest circle covering the triangle as shown in Figure 7.19.

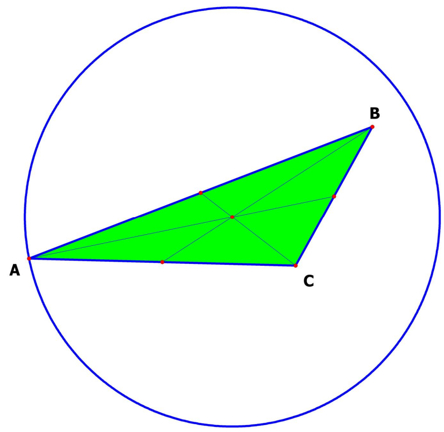


Figure 7.19 Pauline's circle construction from the triangle medians' intersection

Pauline then fixes her strategy by using the midpoint of side AB as the circle's centre instead, which she claims is the smallest circle she can construct to cover the given triangle ABC as shown in Figure 7.20.

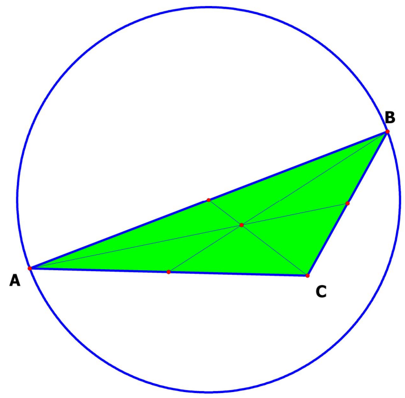


Figure 7.20 Pauline's construction of a circle with AB's midpoint as the centre

When asked for a reason why she thinks that the midpoint AB should be the centre of the circle, Pauline responds by explaining her strategy to find the position of the circle's centre.

Pauline: "The centre of the circle should be at the centre of something. First, I try to construct midpoints and connecting lines which I think should give an intersection; it does give an intersection but this intersection is not the centre of the smallest circle. Then I see that the circle can be smaller if it covers side AB, so I use the midpoint of side AB as the circle's centre. It's not the smallest but I think it is my best answer."

Pauline's assertion that the centre of the circle should be 'at the centre of something' shows her intuitive strategy to solve the problem without clear reference to any known geometric property. It can therefore be deemed abductive reasoning. She poses a conjecture and then tests it and finally opts for the best solution she can find. Pauline's fixation with the idea that the circle's centre should also be a centre of something, and it should lie somewhere within the triangle's area, as she estimated in her initial attempt, prevents her from trying other solutions especially when the circle's centre of this problem actually lies outside the triangle. Nevertheless, her reactions clearly show a systematic strategy to find the circles' centre based on her belief that the position should also be the centre of something.

Peter's response to problem solving task also incorporate abductive reasoning. When asked to construct a new circle touching a given circle O at point A on the circumference, Peter conjectures that the new circle should be the same size with the given circle and its centre aligned to the radius OA. He first uses the Segment tool to construct a radius OA and then tells the researcher that he wants to extend the segment OA out of the circle, for which the researcher advises him to use the Ray command. Peter then uses the Copy and Paste commands to copy and paste the radius OA in order to find the position of the new circle's centre on the extended ray. He then constructs a new circle as shown in Figure 7.21.

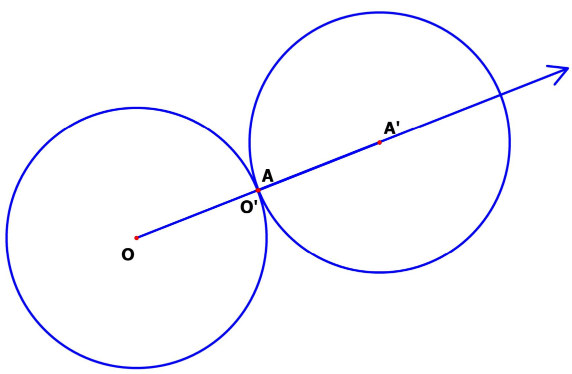


Figure 7.21 Peter's attempt to construct a touching and equal circle

Nevertheless, when he tries to drag-test the construction by dragging point A’, he finds that the new circle’s centre A’ does not really lie on the extended ray. However, he observes from a dummy-locus dragging, i.e. dragging point A’ along the extended ray that the new circle A’ would still touch the given circle O if the centre A’ lying anywhere on the extended ray. Peter then deletes the non-robust circle A’ and uses the Compass tool to construct a new circle with its centre lying on the extended ray. He then demonstrates that the new circle can be bigger or smaller than the given circle, and it can also be inside the given circle by dragging the new circle’s centre along the extended radius as shown in Figure 7.22.

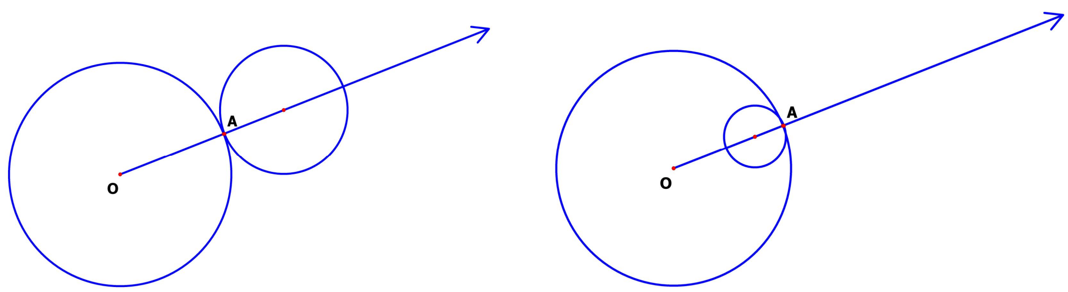


Figure 7.22 Peter demonstrates a range of possible new touching circles

When asked why he first tries to construct a new circle with the same size of the given circle, Peter claims that he saw a diagram of two touching circles of equal size from a textbook previously, so he attempts to construct a figure like that diagram, but he finds out by himself later that the touching circle does not need to be of the same size as the given circle. The researcher then asks him why he tries to extend the radius OA and Peter responds that he just guesses that the new circle’s centre should lay on that extended line. He also uses the Compass tool to construct a new circle to demonstrate that the circle cannot touch the given circle if its centre lies outside the extended ray. This shows Peter’s inductive reasoning strategy by Crucial Experiment in order to verify his solution. The counter-example is used to support the fact that the only way to construct a new touching circle is to place its centre on the extended ray. Though Peter's abduction of the solution may not involve any obvious geometric property, his strategy to solve this problem possibly stems from the skill of visualisation, which can be regarded as an alternative way of geometric verification. He therefore solves the problem by envisioning what the outcome should look like and then devises a process to construct it later.

Different circumstances take place when Rachel and Ruth try to construct a new chord of the same size in that circle. The girls simply use the Copy and Paste commands in order to construct a new chord of the same size of the given chord and then drag the copied segment to the new position as shown in Figure 7.23.

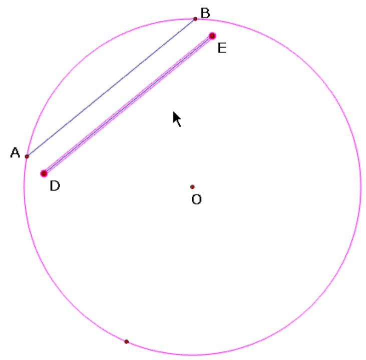


Figure 7.23 Rachel and Ruth's copy and paste strategy

However, the girls realise later when they perform a drag-test that the copied segment does not retain its length when the endpoint is dragged. This behaviour in the GSP amazes them since they expect the copied segment to always share the same length with the original chord. This strategy shows an interesting utilisation scheme of the Copy and Paste commands in the GSP. It highlights the conflict between the users' expectation of the software, with the designer's actual decision involving the interrelationship 3-5-6 in the analysis model. It should be noted that Rachel and Ruth appreciate that the object in the GSP environment is not just a visible object, it also possesses certain properties such as length and orientation. When Copy and Paste commands are used with an object, they therefore anticipate that the GSP will also 'copy' such a property as length and then apply it to the copied figure when the Paste command is executed. However, the Copy and Paste commands in the GSP are designed to duplicate the object only. Other properties such as length or orientation are left independent in order to allow dynamic flexibility. This conflict exemplifies the issue of software design choice since there is no agreed principle of how Copy and Paste commands in the DGS environment are used. Unfortunately, Rachel and Ruth do not find any other strategy to complete this task. The researcher, therefore, cannot elicit more reasoning strategy from them.

Rob and Ryan's response to the problem of constructing a square inscribed in a given circle with a given point as a vertex also shows an interesting strategy. Ryan suggests that they should first construct a straight line through the centre O and the given point A on the circumference in order cut the circle in half, then another line should be drawn to make them look like a cross as illustrated in Figure 7.24.

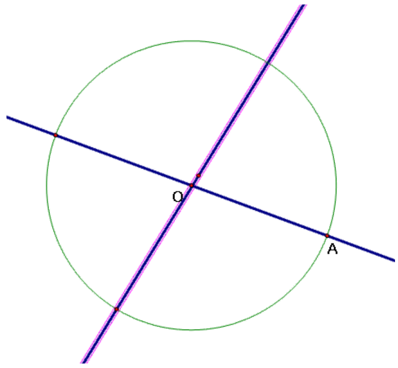


Figure 7.24 Ryan's strategy by halving the circle

Ryan then claims that the intersections of the two cutting lines and the circle's circumference should be of equal distance from each other. Nevertheless, since Ryan draws the second cutting line roughly in order to make the two lines look like a cross, Rob notices that the four intersections do not actually lie apart from each other with equal distance. He then suggests that the second cutting line should be perpendicular to the first line through the centre O. The pair then deletes the second line and reconstructs it with the Perpendicular Line command. They then use the Segment tool to join the four intersections giving a square shape as shown in Figure 7.25.

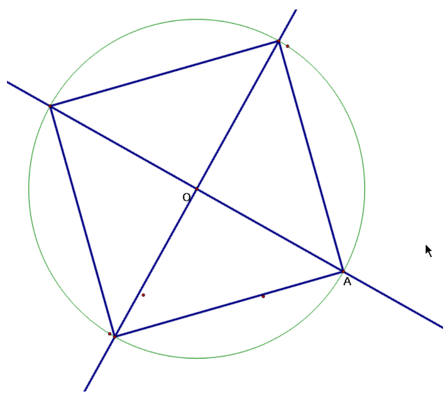


Figure 7.25 Rob and Ryan's construction of a square incribed in the given circle

When asked to verify that their construction is actually a square, Ryan first adopts inductive reasoning to demonstrate with the drag-mode that when any point is dragged the opposite sides of the quadrilateral remain parallel and equal so it should be a square. The researcher then further challenges them to verify why they believe that the opposite sides of the quadrilateral are parallel and equal. Rob then takes turn to explain that the four small triangles to form the square shape are all right angle triangles and the two sides of each triangle are radii of the same circle. The third side of each triangle should therefore have the same length (property of congruent triangles). The researcher then further enquires why they think the four angles of the quadrilateral are all right angle. Ryan responds by explaining that the circle's diameter is the angular bisector of the square and the bisected angle should be 45 degrees, which shows an instance of circular reasoning based on the statement in question (he uses the fact that the quadrilateral is the square as a part of the explanation). When further asked why they think it is the bisector, the boys cannot give other reason but they are certain that the base angles of the small triangles are all 45 degrees.

Rob and Ryan develop a range of reasoning strategies in order to complete and verify their square construction. It is interesting that they use everyday life terms during their process such as 'cut the circle in half', or make it look like 'a cross', rather than adopting mathematical terms such as diameter or perpendicular lines. Their strategy to construct the square seems to be based on their visualisation rather than on geometric property, since they never explicitly mention the property of the square's diagonals (equal, bisect and perpendicular to each other). Nevertheless, they still construct the square successfully. Their reasoning strategies during the justification phase are also varied. They include inductive reasoning by demonstration with the drag-mode, deductive reasoning from the congruent triangles' property and a tautological circular reasoning based on the statement itself.

From the students' responses to the problem-solving task in Task 4, it is evident that reasoning is also involved in both the process of problem-solving itself and during the solution justification phase. Although some students fail to find a solution for some of the tasks, giving the researcher no opportunity to pursue their reasoning in the justification phase, other groups of students show a range of reasoning strategies in order to verify their solution. The designed problem-solving task in Task 4, as well as the task approach, can help the researcher to examine the students' reasoning strategies in the GSP environment.

7.3 REFLECTION ON THE PILOT STUDY AND THE REVISION OF THE DESIGN TASKS

This section discusses how the participant students' reactions to the designed task in the pilot study reflect its suitability as a research instrument to elicit students' reasoning in the GSP environment. It also discusses the research setting to facilitate the researcher investigate the situation more thoroughly. The process of task revisions based on the data gained from the pilot study used in the main study is then outlined.

7.3.1 Reflections on Participant Students and the Research Setting

The choice of Grade-8 students appears to be suitable for this research. Students participating in the pilot study are generally well-equipped with computer skills and have the level of prior knowledge appropriate for the study. All the geometric exploration and problem-solving tasks are also new to them since they are included in Grade-9 lessons, giving the researcher an opportunity to investigate their fresh responses.

Comparing the research settings of interviewing students individually and in pairs show that conducting the interviews in pairs helps the researcher to gain much better insight into the student’s thinking process through their conversions than in the individual interviews. Though the researcher encourages both Peter and Pauline to adopt the thinking aloud method, i.e. verbally explaining every step of their strategy, they seem to find such method uncomfortable and usually think in silence and verbally express only when the researcher asks a question. Peter even requests time to concentrate on the problem by saying that ‘let me think first’. He then tries to work with the figure in silence for a while and finally reports his strategy when he is certain that it works. This contrasts with the case where the students work in pairs, and their thought processes are naturally expressed through verbal discussion. The pair setting also appears to be helpful for students to share their knowledge needed for the tasks. It also promotes argumentation among students, especially when they need to convince or challenge each other when they have different strategies to tackle the tasks. Setting the interview in pairs also helps complete the mathematical argumentation scenario proposed by Mason, Burton et al. (2010), when each student is supposed to convince himself, a friend or the collaborator and then convince the enemy or the sceptic and it is the role of the researcher to challenge the student. It makes the mathematical discussion richer than in the case of an individual interview where the argument takes place between the student and the researcher only.

Nevertheless, the students' responses to the designed task in the pair interviews appear to take the form of student contribution. However, it is still possible for the researcher to track the thinking of each student even when they work in pairs. The researcher also has an opportunity to allow each student to pursue their idea by inviting them to try what they like even though it would not be the agreed solution of the pair. Nevertheless, the main obstacle of conducting research in pairs is the software control where students have to take turns to share the mouse. Though the researcher discovers during the pilot study that two mice can be connected simultaneously to the computer and each student can have his/her own mouse, the students still need to take turns to control the software otherwise the cursor would be uncontrollable from double input. Despite the mouse control sharing, each student still has an opportunity to contribute his/her idea or reasoning even though he/she is not controlling the mouse. Giving both students in the pairing the opportunity to try the GSP commands in Task 1, provides each student with a fair experience of what the software can do, hence he/she can still suggest the action without direct contact with the software.

Pairing students of the same sex seems to make the students feel comfortable in working with their peer. However, since the pilot study does not include the case of mixed-gender pairing. The comparison between these gender arrangements cannot yet be made.

Since the pair interviews appear to be more efficient for the researcher to investigate the students' thinking process during the task-based interview in the GSP environment than in the individual interview, all the eighteen participant students of the main study work in pairs, giving a total of nine pairs of students for the main study phase. In order to vary the pairings to enhance the students' responses, the nine pairs of students are both same-gender and mixed-gender. The nine pairs therefore comprise three pairs of boys, three pairs of girls and three pairs of boy and girl. The student pairs are also categorised by their level of mathematical ability, as assessed by the teacher, in order to enhance the variety of their responses. The three groups of each gender-pair are classified into above-average, average and below average students. Table 7.2 shows the pairings by gender and mathematical achievement of all eighteen participant students referred to by English pseudonyms.

Table 7.2 Pairings of eighteen students in the main study

Students Pairings	Boy-Boy	Girl-Girl	Boy-Girl
Above-average	Alex-Alan	Alice-Alma	Aaron-Anna
Average	Brian-Bruce	Barbara-Beth	Bob-Bridget
Below-average	Colin-Conrad	Carla-Carol	Charles-Chloe

The researcher also finds from the pilot study that leaving students to work with the software by themselves, especially in Tasks 2-4, without the researcher's presence, greatly helps in putting the students at ease and being more adventurous in performing the tasks. This strategy is also maintained in the main study. The researcher returns after the given period of time or when the students give a sign that they find the solution to the tasks or need further help.

7.3.2 Reflections on the Designed Tasks

The structure of the interview tasks which are divided into four stages give a clear flow of the GSP familiarisation, basic utilisation and then the adoption of learned commands to help them reason and solve given problems. The students manage to go through the series of designed tasks using both their prior knowledge of geometry, the introduced commands, as well as their own reasoning strategies. However, the detailed responses from the students in each task provide useful input for the researcher to modify each task for the main study. These are discussed by task as follows.

Reflection on Task 1

From the students' responses in Task 1 to the DG commands exploration in the pilot study, it appears that students need a significant amount of time in order to learn all the selected commands, especially when they have to figure out the commands' algorithm by themselves. In order to manage the activity according to the allocated time of 15 minutes, the researcher decides to further exclude some of the basic commands that do not appear to be useful for the geometric construction and exploration tasks. Two commands in the Display menu: Trace and Erase Traces, and the Angular Bisector command in the Construct menu are excluded from the DG exploration task for the main study.

The data from the pilot study also shows that some students may have the idea of using other commands available in the GSP to help them reason in a later tasks. For example in Task 3, Peter visualises a right isosceles in a semi-circle as half a square, suggesting that the reflection operation may be helpful for his strategy. Nevertheless, the researcher decides to exclude Transformation commands from Task 1 in the main study and concentrates on commands related to Euclidean geometry due to the time constraint. Even though it is apparent from the pilot study that the more commands students know, the broader strategies they may adopt to help them perform the tasks.

The task that instructs the students to use the learned commands to construct any picture they like also helps to reflect the students' appreciation of what GSP can be used for. It is also included in the main study.

Reflection on Task 2

Of the four tasks designed for the Geometric Construction in Task 2, the parallelogram construction task appears to have a level of difficulty suitable for students working in pairs. For the task to construct a robust isosceles and a robust rhombus, though guidance to use the Compass tool is meant to help students, it appears to mislead them, especially when they have a different strategy to construct the figure from what is intended. However, without such guidance, the task to construct a robust isosceles and a rhombus can be even more challenging for students at this level, especially when the Transformation commands are not introduced. The task of constructing an equilateral triangle can also lead to the same problem, since it requires the Compass tool to construct radius-sharing circles to complete the figure without the rotation tool. The parallelogram construction task, therefore, turns out to be the most appropriate task for geometric construction in Task 2. Moreover, the multiple steps of construction required for the parallelogram also provide a more valuable case of the parent-and-child relationship for students to experience, especially where there are multiple levels of relationship. This also helps the researcher to improve the examination of the students' reaction to the parent-and-child relationship in GSP construction.

Reflection on Task 3

The geometric exploration and justification in Task 3 appears to be the task that best helps the researcher to elicit students' reasoning strategies. The task also takes less time that originally planned. Two tasks instead of one can then be included in Task 3 in the main study in order to enrich the situations for students to reason. Since all groups of students manage to discover the intended property with some guidance from the researcher, and manage to give at least some strategies to reason such discovered property in their own way. Any task in Task 3 from the pilot study can be used in the main study. Nevertheless, since the triangle's midpoint theorem and the quadrilateral's midpoint theorem share obvious similarities, as they both involve midpoint construction, only one task of these two are selected for the main study. The triangle's midpoint theorem seems to be more suitable for the students at this level as they have just learned about the concept of similar triangles from past lessons which can help them reason the parallel property. It is therefore included in the main study. In order to provide more variety in the geometric exploration task, the perpendicular bisector of the circle's chord task is selected as another task instead of the triangle in a semi-circle. These two exploration tasks: one about a circle and one about triangles, are expected to elicit broader strategies from the students, especially when different areas of background knowledge are required for the tasks. The researcher maintains the approach to let the students perform additional constructions so they are aware of the construction process, lets them explore the figure with the necessary guidance and then challenges them to reason the claim with their own strategies.

Reflection on Task 4

Task 4 appears to be the most challenging for the students to tackle. The only problem that students manage to find a solution to, and then adopt some deductive reasoning to justify their solution, is the construction of a square inscribed in a circle with a given point on the circumference as a vertex. Three other problems are: constructing a new circle touching a given circle at a given point on the circumference, constructing a new chord of the same size with the given chord in a circle and constructing the smallest circle to cover a given triangle. These can be more challenging and require more advanced knowledge of the circle's property than students have yet learned. The problem of constructing a square inscribed in a circle seems to be the most suitable task for students at this level to construct and then adopt deductive reason, based on the property of the triangle and circle the students have already learned. This task is also interesting because it can be solved with a visualisation strategy, as happens in Rob and Ryan's case. The problem should therefore invite different kinds of strategies to solve and verify and is used in the main study.

Apart from the interview itself, the researcher also plans a Critical Incident Recall interview to be conducted with the participant pairs of students the day after the main study interview for 15 minutes. This is to allow the researcher to follow up interesting reasoning strategies or to clarify possible ambiguous points made by students.

This chapter outlines the process of overall activity design in order to finalise the set of tasks, as well as the research setting to be used in the main study. It concentrates on the design process itself, the analysis of the data gained from the trials with the students in the pilot study, and the reflection and revision of the tasks. The whole process should help answer the research sub-question 3, which looks for the kind of task that would help elicit students' reasoning strategies. The overall assessment of this process is repeated in Chapter 11 where all outcomes from Chapters 7-10 are used to discuss the research findings.

The final set of tasks for the main study, including the instructions, pre-constructed sketches and the expected solutions is presented in Appendix B of this thesis.

8 APPLICATION OF THE ANALYSIS MODEL

With the finalised designed tasks for the task-based interview developed in the previous chapter, the researcher conducts the main study with all nine pairs of students. The data gained from the main study phase includes: video and audio recordings of the screen as well as students' physical interactions with the screen during the task-based interview; GSP files of the performed task; video and audio recording of the critical incident recall interviews; and the researcher's observation notes both during the interview and after reviewing the interview recordings. Segments of significant reactions from students pertaining to the research questions are identified during the review of the recorded interviews. These segments are then fully transcribed by the researcher in order to analyse the relationships based on the analysis model.

This chapter aims to illustrate the application of the analysis model shown in Figure 6.3 in Chapter 6 to selected portions of the data gained from the main study. This is to demonstrate the analysis approach the researcher adopts for the whole data by showing some examples. The section of data to be used in this chapter is the analysis of Task 3.2 where students are expected to explore and reason the midpoint theorem performed by two pairs of students. This particular task is selected because it initiates the widest range of reasoning strategies from the students, and it also implies a certain pattern in students' reasoning strategies. The analyses of the data from two pairs of students among the nine pairs are present in order to illustrate different strategies students can adopt in the same task. The selected pairs are a pair of above-average boys: Alex and Alan and a pair of average girls: Barbara and Beth.

The analysis of this section of data is aimed at answering the research sub-question 2: “What kind of reasoning strategies do learners adopt in the geometric construction and exploration task in the DGS environment?” It reflects the inter-relationship 1-3-5 and 2-3-5 of the analysis model as shown in the highlighted areas in Figure 8.1.

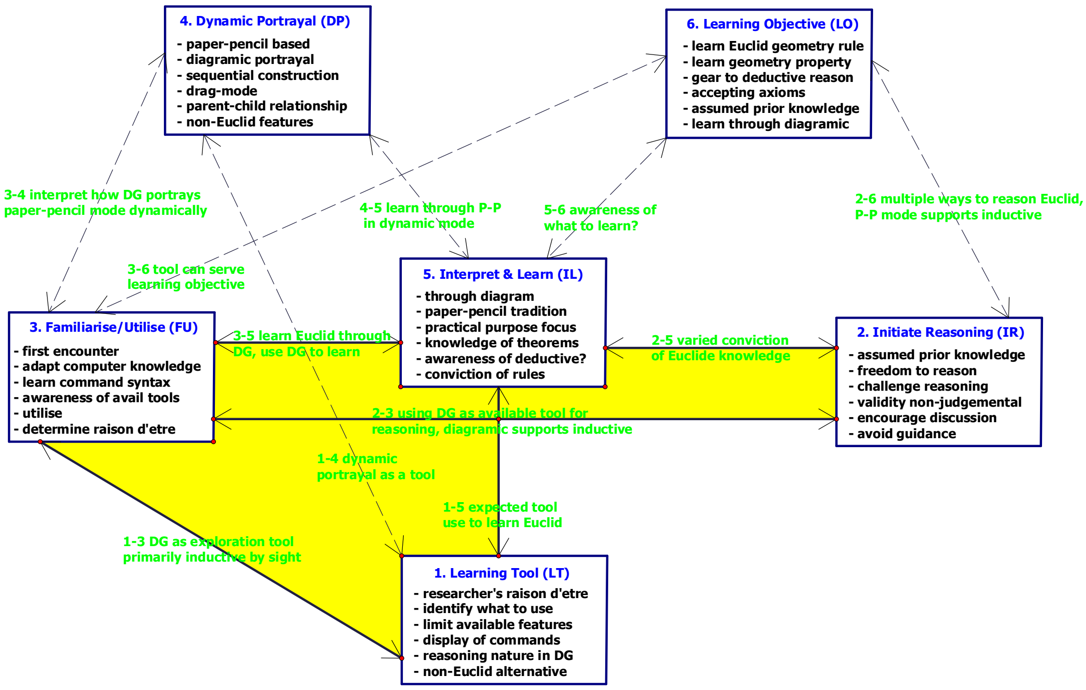


Figure 8.1 Inter-relationships 1-3-5 and 2-3-5

However, the analysis of this section of data also concerns possible interactions with other parts of the model in order see the overall relationships between relevant entities. Apart from the analysis model, the axial codes identified in Section 5.7 are also used to examine students' reasoning strategies in the interview tasks. These codes help the researcher to identify the actions that relate to the research questions based on the relevant literature reviewed in Chapter 2.

This chapter is divided into two sections. The first section discusses the rationale of this particular task to provide its objective, followed by the analysis section of the two pairs of students' strategies.

8.1 RATIONALE FOR THE DESIGN OF TASK 3.2

The instruction for the Task 3.2 is given as follows.

TASK 3.2

Instruction: Construct midpoints for sides AB and BC and label them D and E respectively. Draw the segment DE then move the figure and report your observation.

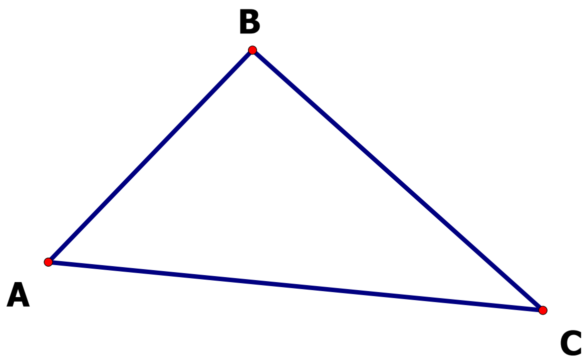


Figure 8.2 Task 3.2

In this task, students are supposed to observe that the segment DE is always parallel to side AC and are challenged to give the reason why.

The first inter-relationship in the analysis model to be discussed is the inter-relationship 2-5-6 where the task is supposed to challenge students' reasoning towards deduction to enable them to learn a new Euclidean geometry rule. The geometry rule selected for Task 3.2 is the triangle's midpoint theorem which is a version of Proposition 2 in *Elements* Book 6 where students are expected to observe that the segment connecting two midpoints of the triangle's two sides is always parallel to the remaining side. Once they discover this property, the students are challenged to give the reason, with the aim of orienting them towards the more rigorous explanation by deductive reasoning. One possible deduction is to use the property of similar triangles by identifying the equal proportions between the two sides of similar triangles and pointing out a common angle. The inter-relationship 2-5-6 in this case is, therefore, letting students learn the midpoint theorem through the process of reasoning under the challenge placed by the researcher with DGS as an available tool.

Though DGS is expected to help students give reason using their own strategies, the students have the opportunity to use just the commands they are introduced to during Task 1.1. This reflects the tension presented in the inter-relationships 1-3-5 and 1-4-5 where students are expected to use commands/features as well as the dynamic functions in DGS to help them in the learning process. In Task 1.1 only some commands in the toolbox, Edit, Display and Construct menus are introduced to the students. This already limits the possibility of the students using other available features such as the measurements or transformation functions to help them reason, therefore restricting them to Euclidean geometry-based approaches of reasoning.

The expectation of students' acquisition of knowledge about triangle's midpoint theorem through the process of reasoning (inter-relationship 2-5-6) is already framed by the fact that DGS is selected as a learning tool (inter-relationships 1-3-5 and 1-4-5) with the deductive reasoning set as the most rigorous type of reasoning (inter-relationship 2-3-5) which the researcher would challenge the students to adopt it.

8.2 ANALYSES OF THE SELECTED DATA

The data gained from two pairs of students: Alex and Alan and Barbara and Beth are analysed by first using the axial codes identified in Section 5.7 to identify students' reasoning strategy. Then the analysis model is used as a framework to examine the inter-relationships between the relevant entities in the students' reasoning process.

8.2.1 Alex and Alan on Task 3.2

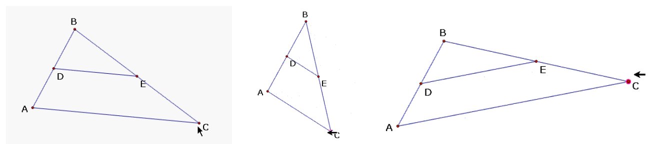
Alex and Alan are classified by the teacher as above-average students. Both students show strong mathematical knowledge and skills and perform the interview tasks relatively well. The analysis shows that students can map their mathematical understanding to the way DGS commands behave quite naturally and they can learn to utilise the tools to help them solve the problems effectively. Though Alex takes more control of the software, Alan always contributes his ideas and opinions and always follows what Alex does on screen so he can also perform the task along with him. These students are very keen to learn to use the software and respond eagerly to any challenge placed upon them.

The following excerpts show their reasoning strategies during Task 3.2. They are presented in sequential order to show their steps of reasoning and are identified by the different processes of reasoning they use to justify the discovered property.

EXCERPT 8.1: Inductive Reasoning by Sight

When Alex and Alan first observe that the segment DE maybe parallel to AC, they show different levels of confidence in justification by sight. The following excerpt shows Alex and Alan's initial response to the task.

Table 8.1 Inductive reasoning by sight

Line	Dialogues	Actions	Reasoning/Comments	Screen
		After Alex and Alan finish the construction, they are asked to move the figure and observe the result.		
	Researcher: Please try to move the figure and tell me what you see. Alex: Move? Where? Alan: Anywhere, just try.			
	(6-sec pause)	Alex moves the arrow tool to the vertex C (a) and then drags the point downward (b) and to the right (c).	Though Alex seems to move point C randomly, it is notable that he does not move point C leftward beyond point A.	 (a) → (b) → (c)
	Alan: The lines remain parallel. Researcher: Which lines? Alan: DE and AC. Researcher: DE and AC parallel?			

Line	Dialogues	Actions	Reasoning/Comments	Screen
		Alan keeps moving point C around.		
	Alan: Yes . . . or not? (reluctant) Alex: Parallel! (with confidence) Researcher: Are they parallel? Alex: Always parallel. For sure! Alan: Maybe not. It’s just by our eyes.		Alex is convinced by inductive reasoning from what he sees. Alan expresses his lack of confidence in inductive reasoning by sight.	

Though Alan first notices that DE and AC are parallel to each other after a brief drag-test (Line 14), he later shows uncertainty by explicitly claiming that justification by eyesight may not be reliable (Line 24). In this circumstance, drag-mode in DGS is used to test if segments DE and AC are truly parallel especially when the figure is moved around. This seems to convince Alex, since he can see that these segments remain parallel regardless of how the triangle's appearance is changed, while Alan's response suggests that justification only by sight is not sufficient for the conclusion. Nevertheless, his response implies that he half-heartedly believes that they are truly parallel (Line 24).

Alex's confidence in the observed property seems to stem from a combination of two observations: 1) DE appears to be parallel to AC and 2) this property is retained when the triangle is moved. The first part of the observation can be regarded as the data while the dynamic observation in the second part can be considered a warrant, i.e. the property does not change when the triangles' component is moved.

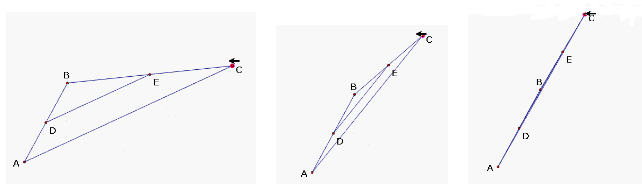
Nevertheless, Alan argues that observation by sight cannot be mathematically reliable despite the obvious support from the dynamic invariance. The fundamental argument between Alex and Alan in this excerpt is whether the induction merely by observation is mathematically acceptable to verify a claim.

In this excerpt, the dynamic feature in the GSP plays a role in helping Alex to ascertain that DE is always parallel to AC. It provides a different form of inductive reasoning, where the change of the same figure provides a multiple set of data that students can generalise rather than using different figures. This utilisation scheme of drag-mode to verify geometric property illustrates the relationship 1-3 in the analysis model presented in Figure 8.1, where DG is used as an exploration tool providing concrete evidence for induction by sight and students may affirm their observation as a geometric rule, as is shown in the relationship 3-5. However, Alan does not accept that such relationships would lead to a valid conclusion. This implies that he has a natural doubt about the validation based purely on perception even in a computerised environment such as in the GSP, and demands a better form of reasoning in order to convince him.

EXCERPT 8.2: Abduction by Collapsed Figure

After Alan voices his doubt on the inductive justification by sight, Alex tries to convince Alan by using the dynamic feature to adjust the triangle until all the segments collapsed or aligned which Alex assumes to be a verification of the parallel property of DE and AC.

Table 8.2 Abduction by collapsed figure

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Alex: Always parallel because of this.	Alex shows that DE and AC are parallel by dragging point C (a-b) until the triangle is collapsed (c).		 (a) → (b) → (c)
	Researcher: What are you trying to do?			
	Alex: To prove that they are truly parallel.		Alex uses the word 'prove' for his demonstration.	
	Researcher: So why do you move the figure that way?			
	Alex: Because if they are truly parallel, when you drag to this point they will be the same line.		Alex abductively maps the parallel property with his observation when the figure is collapsed.	
	Alan: Wait, but BC and AC are not parallel, they still . . .		Alan uses counter-example to challenge but Alex argues that the two cases are different.	
	Alex: No, I mean they overlap with each other. These (BC and AC) are not overlapped.			

In order to show that DE is actually parallel to AC, Alex guide-drags point C until the triangle is collapsed and DE is overlapped with AC. He claims that if two lines are parallel with each other, they become the same line when they are dragged towards each other. From the Critical Incident Recall interview, Alex further explains his strategy. He refers to the known property of parallel lines which must be two lines that never cross. When he tries to drag the figure until DE and AC are overlapped, DE and AC never cross each other. Therefore, DE and AC are truly parallel. This reasoning strategy can be categorised as an abductive reasoning where a known geometric rule of parallel lines is selected to explain why DE and AC do not cross each other when the figure is collapsed. Alex's basic strategy is, therefore, connecting the 'never cross' property of the parallel lines and the observed figure and assuming that they are related, though Alan's argument clearly shows that it is not the case.

Alex's utilisation of the GSP's drag-mode to guide-drag the figure to abductively verify the parallel property shows his appreciation of dynamic environment's flexibility allowing him to reason from a special case i.e. the collapsed figure in this excerpt. The dynamic feature therefore helps Alex abductively reasons from an observable instance with an assumption that if the property in the collapsed figure is true the statement is also true in other cases. This reflects the relationship 4-5 in the analysis model where the dynamic feature is used to help student learn geometry rule provided that DG is chosen as the learning tool (relationship 1-4).

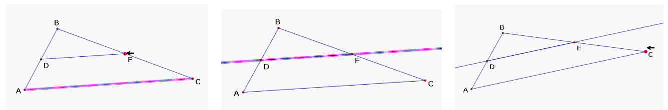
However, Alex's abductive reasoning strategy from the collapsed figure is not adopted to convince him. Alex appears to be strongly convinced that DE is parallel to AC by inductive reasoning by sight discussed in the first stage. He adopts this strategy in order to convince Alan and possibly the researcher, reflecting the move of argumentation level from convincing oneself to convincing a friend and/or enemy as proposed by Mason, Burton et al. (2010). Setting the students to work in pairs therefore helps in enriching the reasoning strategies where students need to convince multiple parties before the claim is mutually accepted. Note that Alex uses the word 'prove' in Line 7 to declare that he has a strategy to verify the parallel property. However, his subsequent actions seem to show that he uses this particular word in an everyday language sense rather than mathematical. The word 'prove' in Thai is commonly used to mean 'verify' in a general sense, not necessarily a deductive verification as defined by mathematicians.

Nevertheless, Alan finds a pitfall in Alex's justification by collapsed figure. He inductively uses a counter-example to point out that segments BC and AC are obviously not parallel in the first place. However, they also appear overlapped when the figure is collapsed. This argument reflects the property discrimination process of inductive reasoning as proposed by Klauer (1996), where the perfect alignment of two lines when they are collapsed is not an exclusive property of parallel lines. Alex defends this by arguing that BC and AC are not exactly overlapped like DE and AC, possibly from the fact that the label of point B appears on the left side of the collapsed figure while D and E are on the right side (figure c in Table 8.2) giving an impression that BC is close but not overlapped with AC. However, since this justification strategy is partly based on visual observation, the students cannot logically conclude whether BC and AC are actually overlapped or not. These different approaches in reasoning to discuss the geometric circumstance reflect the relationship 2-6 in the analysis model where students are supposed to use various strategies to learn Euclidean geometry in the DG environment which is set as one of the learning objectives.

EXCERPT 8.3: Justification by Checking with a Construction Command

Despite the above strategies, Alan seems to be uncertain that DE is actually parallel. Alex then proposes another way to show that they are parallel by using the construction command in DGS as a checking tool. The following excerpt illustrates this strategy.

Table 8.3 Justification by checking with a construction command

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Researcher: So are they parallel?			
	Alan: They should be parallel.		Alan is more convinced.	
	Researcher: Did we construct them to be parallel?			
	Alan: No, we use the midpoints.			
	Researcher: So why are they parallel?			
	Alex: In order to prove . . .			
		Alex uses parallel line command (b) to check if the DE and AC are parallel then drags point C upward to test (c).		 (a) → (b) → (c)
	Alan: Exactly.		Alan is convinced immediately after the parallel line is constructed.	
	Researcher: How did you prove?			

Line	Dialogues	Actions	Reasoning/Comments	Screen
20	Alex: We construct a parallel line. Alan: And they are parallel.	Alex hides the parallel line.	Alan concludes inductively by sight.	

While discussing whether DE and AC are truly parallel to each other, and while trying to find an explanation, Alex proposes another strategy to use the Parallel Line Construction command to check if the segments are actually parallel. Once Alex completes the command to construct a parallel line with the base AC through point E (Lines 8-12) and they see that the constructed parallel line is perfectly aligned with the segment DE, Alan is convinced immediately that this should confirm that DE is actually parallel to AC as he exclaims 'exactly!' in Line 13. Despite Alan's explicit conviction, Alex moves on to drag test by moving point C in order to see whether the constructed parallel line remains on DE (Lines 11-12). This action shows Alex's awareness of the importance of drag-test in the DGS environment where any property should be tested for invariance with the drag-mode.

The use of the Parallel Line Construction command as a checking tool demonstrates the students' own utilisation scheme of the software command in order to verify their observation, which may differ from the designer's intention of the command as a construction tool. Unlike other DG products such as Cabri-Geometry which has a dedicated menu for geometric property checking features such as parallel, perpendicular or collinear, the GSP does not include such features. Nevertheless, Alex manages to creatively use the Parallel Line command in the Construct menu to perform this similar task, though he and Alan need to rely on visual observation in order to affirm that the lines are truly aligned by using the drag-mode. This strategy can be analysed as a combination of two different types of reasoning. The first is deductive reasoning from the reliability of the software command, which should generate true parallel lines and can be used as a yardstick. The warrant for this part of reasoning is therefore the accuracy of software construction. The second reasoning strategy is inductive reasoning by sight, where the constructed parallel line needs to be observed as being perfectly overlapped with DE. The dynamic feature by the drag-mode is also used to inductively verify the parallel property, similar to the case in Excerpt 8.1.

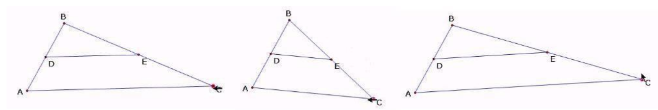
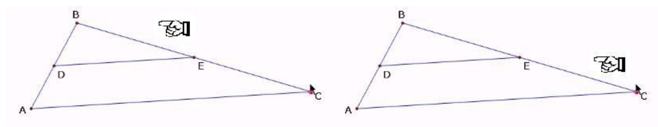
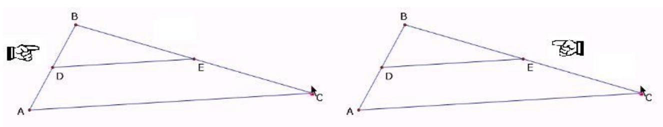
This circumstance clearly relates to the relationship 2-3 in the analysis model where students are supposed to use DG features to help them reason in the interview task. The DG features used in this excerpt is the Parallel Line Construction command and the drag-feature which are used in combination to verify that DE and AC are parallel to each other.

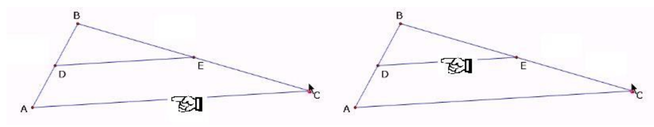
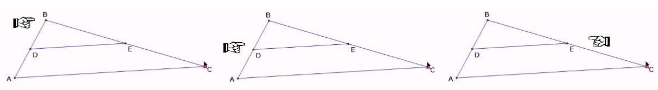
It is notable that this is the first time Alan is undoubtedly convinced that DE is truly parallel to AC, even though he has to rely on sight to ascertain that the parallel line command actually overlaps the segment DE. This maybe the result of repetitive justification of the same claim by different strategies as Alex tries to convince him during the first two excerpts. When the third strategy also shows the same result, it is sufficient for Alan to believe that the claim is true.

EXCERPT 8.4: Abductive Reasoning and Deductive Reasoning towards the property of similar triangles.

After the researcher leaves Alex and Alan to discuss and discover the supporting reason themselves, Alan points out that the midpoints actually half the triangle's sides. This observation leads Alex to abductively conjecture that the parallel property may relate to the concept of similar triangles for which they try to adopt deductive reasoning to explain later. The following excerpt shows these responses.

Table 8.4 Abductive reasoning and deductive reasoning towards the property of similar triangles

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Researcher: Okay, can you explain why DE and AC are parallel since we didn't construct them to be parallel. We just connected the midpoints D and E.			
	Alex: Yeah, that's true.		Alex is aware of the consequence from the construction.	
	Researcher: Please find the reason why.			
		(Researcher leaves) Alex moves point C around briefly then stops (figures a-c).		 (a) → (b) → (c)
	(5-sec pause)			
	Alan: This (BE-figure d) is half of this (BC-figure e), right?	Alan uses his finger to points at the screen (figures d-e)	Alex abductively reasons that this may be the rule of similar triangle.	 (d) → (e)
	Alex: Ah! Similar triangle!			
	Alan: Mmm (convinced). But how to explain that they are similar.		Alan is aware that more reasoning is needed.	 (f) → (g)
	Alex: Let's try. When we construct the midpoints. This (AB) is doubled the	Alex now uses his finger to point at the screen		

Dialogues	Actions	Reasoning/Comments	Screen
length of this (BD-figure f) and this (AC) is doubled the length of this (BE-figure g)	without using the mouse (figures f-g).		
Alan: And . . . ?			
Alex: And they are similar (laughs shyly) (4-sec pause)		Alex is aware that his reasoning is illogical.	 (h) → (i)
Alex: So, this (AC-figure h) is also doubled the size of (DE-figure i), then they are parallel. Nah, it's not relevant.		Alex and Alan are aware that their deductive reasoning does not make sense.	
Alan: Yeah, not relevant at all.			
	(Researcher returns)		
Researcher: So why do you think they are parallel to each other?			
Alex: These two triangles are similar triangles.			
Researcher: Why are they similar?			 (j) (k) (l)
Alex: Because these angles (B-figure j) are equal, a common angle. And BDE equals to BAC (figure k) and BED equals	Alex uses his finger to point on screen.	Alex refers to the common angle for the first time.	

Line	Dialogues	Actions	Reasoning/Comments	Screen
40	to BCA (figure I). Researcher: Why are they equal? Alex: Because for parallel lines, these two angles (BDE and BAC) must be equal. Researcher: But we are yet to confirm that they are parallel? Alex: Oh, yes.		Alex uses circular reasoning to verify the property but realises that he uses the claim for the explanation.	

Just when Alan starts to collect the figure's property by citing that BE is half the length of BC, Alex realises immediately that the parallel property might be a consequence of a similar triangle case (Line 13). This can be considered an abductive reasoning since there is yet no logical connection between the parallel property and the concept of similar triangles. Alan, therefore, expresses that more reasons are needed in order to confirm that the triangles are similar (Lines 14-15). This response shows that Alan is aware that abductive reasoning simply gives a conjecture and is not sufficient to validate the claim. Nevertheless, Alex's identification of the similar triangle sheds light on a new direction which Alan eventually follows.

Once the concept of similar triangles is raised by Alex, he then tries to map the given property from the construction, i.e. the fact that D and E are midpoints of sides AB and BC respectively, to verify that ABC and BDE are similar triangles. He first points out the double ratio of segments DB-AB and BE-BC as the sole explanation (Lines 17-19) but he realises later that his reasoning is insufficient and therefore not logical (Lines 19-20). Nevertheless, the structure of Alex's explanation when referring to the known property of the segments' ratio, and then using it to conclude that the triangles are similar, still reflects the form of deductive reasoning. The only problem is that the statements he chooses as the premises are not logically sufficient to support his claim. The strategy is therefore rejected by Alan (Line 26). Alex's attempt to adopt this particular form of reasoning suggests that he values deductive reasoning as a valid form of reasoning to be used to convince others, though he realises that the content of the reasoning should make sense too. Note that the word 'relevant' both students use in Lines 25-26 suggests that Alex's reasoning does not make sense and should not be relevant to the valid approach of verification. They do not seem to adopt the literal meaning of the word that the statements are not entirely related to the claim they are trying to verify.

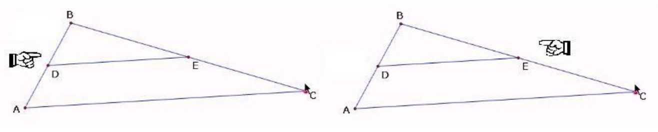
Surprisingly, Alex refers to the common angle ABC and DBC for the first time after the researcher returns but he never mentioned this property with Alan during their discussion. Alex's reasoning strategy then changes from using the ratios to the property of the parallel lines' corresponding angles, though he later realises than he actually adopts a circular reasoning (Lines 38-43). In fact, Alex and Alan's discussions bring about a range of given and observed properties that they may use for a successful deductive reasoning. Nevertheless, the two boys fail to connect the double ratios of segments DB-AB and BE-BC with the common angles ABC-DBC in order to verify that ABC and DBE are similar triangles. Nevertheless, they attempt to use deductive reasoning to verify their claim based on known properties which seems to be the reasoning strategy students find most rigorous to explain the observed geometric relationship in the figure.

It should be noted that once the students start to move on from inductive reasoning to abductive and deductive reasoning, the dynamic feature in DGS is hardly used. As can be seen from this excerpt, students discuss their ideas based on the static figure without moving anything further. The construction of the GSP in this situation provides similar resources to those in the paper-and-pencil environment since the dynamic or other exclusive features in the GSP is hardly used to support the reasoning. These reasoning activities therefore, mainly involve the inter-relationship 2-5-6 in the analysis model, where the design of the task is aimed at initiating the students' reasoning strategy with little use of the DGS features. It shows that the designed task in this task succeeds in eliciting the student's thinking process of reasoning.

EXCERPT 8.5: Abduction with Transformational Reasoning

After failing to fully explain the parallel property with the concept of similar triangles, Alan adopts transformational reasoning to explain the parallel property of the figure. He describes the way each side of the triangle is contracted with the same ratio, keeping DE and AC parallel to each other. The following excerpt shows his response.

Table 8.5 Abduction with transformational reasoning

Line	Dialogues	Action	Reasoning/Comments	Screen
	Alex: Oh, they are similar triangles because the ratio of BD and BA is equal to the ratio of BE and BC. Researcher: Why are they equal? Alan: What I think is BD is contracted by half of BA (figure a) and BE is contracted by half of BC (figure b). They are contracted by the same ratio so they remain parallel. If we contracted both by 60 percent, they should also be parallel.	Students use their finger to point at the figure on screen without moving it.	Alan abductively reasons that the parallel property is the consequence of equal ratio contraction based on transformational reasoning.	 (a) → (b)

While Alex attempts to justify the property of similar triangles by deduction, Alan's strategy relies more on transformational reasoning when he applies a dynamic rule to explain the observation. Alan adopts the concept of contraction with 1:2 ratio to explain why DE is parallel to AC (Lines 5-7). He also elaborates later that the contraction can be in any ratio and as long as the contraction of both sides, AB and BC is equal (Lines 9-10), the segment DE would still be parallel to the base AC.

With the word 'contract', it suggests that Alan visualises the case as a continuous action where the base AC is dilated towards point B with equal ratios, i.e. DE is a result of AC after it is dilated rather than a separate entity. The fact that Alan also points out that the ratio is not necessarily half may imply that he is aware of the other possibility that DE is parallel to AC, especially when D and E are not midpoints of AB and BC. He therefore generalises the case to be 'any' ratio as long as they are equal. Nevertheless, Alan does not fully explain why when sides AB and BC are contracted with the same ratio, DE would be parallel to AC. The reasoning remains at the abductive level. Alan's use of transformational reasoning can be a consequence of his experience in the DGS environment when things can be manipulated dynamically, guiding him to envision the static shape appearing on the GSP screen as a result of a dynamic process.

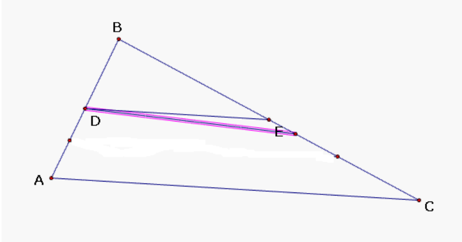
Though it is obvious that Alan refers to the concept of dilation in this explanation of the figure's behaviour, the fact that the set of commands in the Transformation menu in GSP are not introduced to the students in Task 1.1, may prevent Alan from adopting the feature to help him reason. This reflects the tension in the inter-relationship 1-3-5 where the choice of commands decided by the researcher may influence or limit the means students may use in their reasoning process. The Transformation menu is excluded from Task 1.1, mainly because it belongs to a different domain of geometry which may not directly relate to Euclidean geometry reasoning. However, from the findings of this research, it appears that students may opt for different domains of geometry to explain the Euclidean geometry situations they encounter especially when working in the DGS environment, where different kinds of transformation always takes place. Nevertheless, Alan manages to verbally explain his idea without using the Dilate command in the GSP to demonstrate.

Though Alan does not use any feature in the GSP to help him reason in this extract, his reasoning clearly shows the dynamic nature of thinking. This can be evidence of an internalisation of dynamic environment in his process of thinking; by visualising a triangle contracted in a constant ratio with one vertex as a point of dilation resulting in an invariant orientation of that point's opposite side.

EXCERPT 8.6: Deductive Reasoning by Modus Tollens

In order to support Alan's transformational reasoning, Alex uses deductive reasoning by modus tollens by showing an example of a converse case to the researcher.

Table 8.6 Deductive reasoning by modus tollens

Line	Dialogues	Action	Reasoning/Comments	Screen
5	Alex: Yeah, if they are not contracted equally like this, they won't be parallel.	Alex constructs another segment to show the contraction by different ratios (figure a).	Alex deductively reasons by modus tollens which is a converse case of the claim about equal ratio contraction to support their justification.	 (a)

Besides adopting abductive reasoning to explain the concept of contraction with equal ratio, Alex supports Alan by using a mathematical argument of ‘modus tollens’, or presenting the statement in a converse form to further justify Alan’s claim. He demonstrates that DE would no longer be parallel to AC if point E is not the midpoint of BC, i.e. the ratio of contraction is not always 50 per cent. Note that Alex’s adoption of modus tollens does not genuinely provide further data in the reasoning process. He simply infers from the assumed property that if the triangle ABC is contracted with a constant ratio into DBE, DE would remain parallel to the side AC but using its converse form. Such inference can be categorised as a deductive reasoning based on a known property and even though a single instance is used by Alex to demonstrate the contrary case, he appears to intend it as a representation of a class of figure where the contraction is not uniform. Such generalisation therefore implies deductive reasoning rather than inductive reasoning where multiple individual cases are used for a conclusion. Though the modus tollens approach in reasoning may not provide further evidence for the justification process, it can still be a useful tool for the students to ascertain the statement by confirming that the converse of the case is also true.

Though the construction command in DGS is used in the case of modus tollens reasoning, the purpose of the construction is simply to demonstrate rather than incorporate it in their reasoning process. This way of demonstration can also take place easily in the static paper-and-pencil environment, and therefore it is not unique in DGS. This deduction by modus tollens therefore does not directly involve the DGS feature and involves only the inter-relationship 2-5-6 of the analysis model.

ANALYSIS OF ALEX AND ALAN'S RESPONSES TO TASK 3.2

With the analysis model presented in Figure 8.1, with the inter-relationships 1-3-5 and 2-3-5 as the focus for this section of data analysis, possible relationships among the pertaining entities are identified to help the researcher examine the question from the students' response. Based on Alex and Alan's performance of Task 3.2, it can be seen that the designed task initiates a range of reasoning strategies from this pair of students and strengthens the relationship 2-5 where students are challenged to reason in order to verify discovered geometric property. Though the approach of the task and the researcher's role as a reluctant believer are originally set as the main challenge for the students' reasoning, it is obvious that letting the students work in pairs also generates a circumstance where students need to apply a number of reasoning strategies in order to convince his/her peer. Alex's persistent attempt to convince Alan in this task already demonstrates this. The reasoning activity is therefore not just an individual cognitive activity one uses to verify a truth, but it also has a social dimension where mutual acceptance is needed before a certain claim can be truly verified. Alex and Alan's different responses to inductive verification by sight may be used to illustrate this point.

With such reasoning challenges from the task, both from the researcher and the collaborating peer, students are encouraged to adopt the appropriate reasoning approach to satisfy such challenge, with the GSP as an available tool they can use to support their reasoning. According to Alex and Alan's responses, a number of DGS features are utilised to help them verify the observed parallel property. This reinforces the inter-relationship 1-3-5 where DGS is actually used as a learning tool to verify the Euclidean geometry rule in a unique way. The DGS features utilised by Alex and Alan include the drag-mode to inductively verify the invariant parallel property between segments DE and AC; the dynamic property of the figure in the DGS environment in order to adjust the shape into a particular circumstance to facilitate reasoning; the construction feature as a checking tool; and a construction tool to demonstrate a modus tollens case. Apart from these direct manipulations of DGS features for their reasoning, Alan's strategy to use transformational reasoning to explain the parallel property by the constant ratio contraction also suggests an internalisation of the geometric figure's dynamic behaviour found in the DG environment in his thinking process. Though it is not clear whether Alan would adopt similar reasoning in the paper-and-pencil environment, his verification strategy obviously incorporates the dynamic process uncommon in traditional Euclidean geometry. This instance may exemplify the influence of the DGS as a learning environment on students' cognitive thinking.

Nevertheless, no unique DGS feature is used in a particular approach of reasoning by this pair of students especially in the deductive reasoning stage of verification, where students stop using the mouse to control the figure, and physically use their fingers to point at the figure on the screen while they are discussing. The lack of DG use for a certain type of reasoning: deductive reasoning in this case, implies the benefit of DG merely for particular types of reasoning, e.g. inductive and abductive reasoning. This may stem from the fact that DGS portrays Euclidean geometry through the concrete visual diagram which lends itself to inductive and abductive types of reasoning requiring concrete evidence. The students then need to switch to a different strategy by drawing from their prior geometry knowledge in order to deductively verify their observation.

Regarding the Euclidean geometry learning action presented in relationship 5-6 of the analysis model, the students appear to learn the intended Euclidean geometry rule of the triangle's midpoint theorem at the point when they are absolutely convinced that the parallel property is always true and takes place at different stages for this pair of students. The efficiency of the DG tool to display the geometric situation accurately, i.e. showing that DE is parallel to AC, regardless of how the triangle's shape is changed, appears to help students observe and learn this rule. Nevertheless, the approach of the task and the challenge from the researcher, encourages the students to learn to reason rigorously before accepting the rule portrayed in the inter-relationship 2-5-6. This demonstrates the importance of the external challenge from the task to initiate better reasoning strategies from the students to avoid them stopping once they are personally convinced of the observed rule.

Alex and Alan's response to Task 3.2 analysed in this sub-section demonstrates how the relevant relationships among the entities influence each other, especially the way the designed task challenges the students to reason and learn the Euclidean geometry rule using the DG as a learning tool. This, in turn, reflects the role DG plays in the overall process of students' geometric reasoning under the track of the designed tasks.

8.2.2 Barbara and Beth on Task 3.2

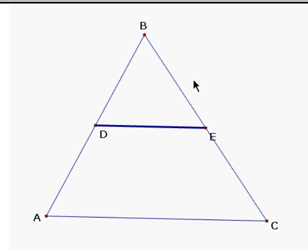
Barbara and Beth are classified by the teacher as average students. They happen to be very close friends who always sit together in the class. The pair persistently collaborates in order to complete the tasks. They seem relaxed in expressing their opinions and suggestions from their close relationship. Though their mathematical knowledge is not always precise, these girls' strategies in solving the task are mainly based on geometric spatial sense rather than rules or theories. Nevertheless, when challenged to refine their reasoning, the girls can also refer to relevant knowledge in order to give deductive reasoning.

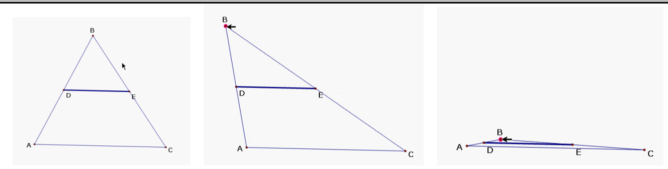
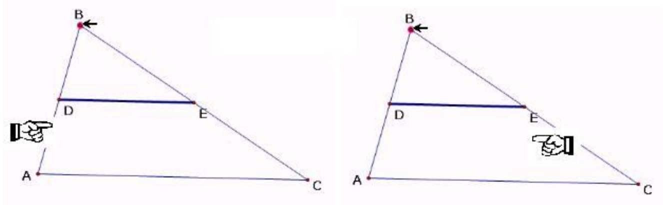
The following excerpts are presented in sequential order in order to show the steps of reasoning. Individual excerpts are identified by the different processes of reasoning the students adopt during the task.

EXCERPT 8.7: Dragging the Figure to Check the Parallel Property

When Barbara and Beth are asked by the researcher to explore the constructed figure by dragging each part, the students use the arrow tool to drag each point and each segment and then describe the way the triangle changes, i.e. which part is moved or which part remains in the same place. They do not observe that DE is parallel to AC on their own. However, once the researcher asks them to pay attention to the segment DE, Barbara spots immediately, without moving the figure, that it is parallel to AC though she and Beth later express uncertainty. Both students then adjust the figure in order to see closely if DE is parallel to AC.

Table 8.7 Dragging the figure to check the parallel property

Line	Dialogues	Actions	Reasoning/Comments	Screen
		After the students drag the figure and describe all the changes, they stop dragging where the figure appears as shown.		 (a)
	Researcher: Do you notice anything else?			
	(3-sec pause)			
	Beth: No.			
	(5-sec pause)			
	Researcher: How about segment DE?			
	(4-sec pause)			
	Barbara: It is parallel to AC, or not?		Barbara states her conjecture but then hesitates.	
	Beth: Parallel?			
	Barbara: Is it parallel?			
		Barbara moves an arrow tool to drag point B	Barbara inductively changes the distances	

Line	Dialogues	Actions	Reasoning/Comments	Screen
		leftward (figure c) then back and down to AC (figure d) then up again.	between DE and AC to check the property.	 (b) → (c) → (d)
	Barbara: Is it parallel? Both: Parallel! (confidently) Researcher: Why do you drag that down? Barbara: To look when they are closer. Beth: Compare.			
	Barbara: To see if this side (distance between DE and AC on the left-figure e) and this side (distance between DE and AC on the right-figure f), the distances are equal or not.	Barbara physically uses her finger to point the figure on screen.		 (e) → (f)
	Beth: They overlap. Researcher: Did we make them parallel? Beth: No, we constructed the midpoint for AB as point D.		Beth remarks that DE and AC are overlapped when they are dragged close together.	

After guidance from the researcher to observe the segment DE, Barbara notices that DE may be parallel to AC though she hesitates to claim this especially when Beth questions her (Lines 13-15). Barbara then uses the drag-mode in the GSP to test the parallel property first by moving point B to the left, changing the appearance of the triangle ABC from an acute triangle to an obtuse triangle (Line 18, figure c). This can be interpreted as guided dragging, used to adjust the figure into a different category of the same shape to verify the observation. After that she drags point B downward until DE is close to AC in order to inductively examine whether they are still parallel (Lines 16-19, figure d). This dragging strategy has a different purpose from when she drags point B since the adjustment is intended to facilitate observation by sight rather than to change the appearance of the figure. Barbara's utilisation of the drag-mode to help her inductively verify the parallel property in this case implies her impression of the flexibility of the shape's appearance in the GSP environment. The triangle is now defined as a geometric shape with three sides, where the appearance can be adjusted in multiple ways while preserving its inherent property. This impression involves the inter-relationship 1-4-5 in the analysis model where the dynamic feature in DGS is used as a learning tool to help students reason. Such flexibility is unique in the DGS environment and not present in the paper-and-pencil environment.

When asked to explain her strategy, Barbara gives a reason for this action by expressing that she wants to move DE down as close to AC as possible so it is easier for her to observe the distance between segments DE and AC both at the left and right side of the triangle (Lines 26-31, figures e and f). Her reference to the observation criterion, whether the distance between segments DE and AC remains the same for both the left and right side of the figure suggests a deductive reasoning strategy where the fundamental property of the parallel lines is explicitly cited. Barbara's strategy therefore involves not only inductive reasoning where she simply observes by eye whether the segments are parallel or not. She also deductively adopts the property of the parallel lines she knows as a criterion to strengthen her claim, though the property of the constant distance is still verified by observation.

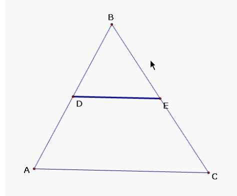
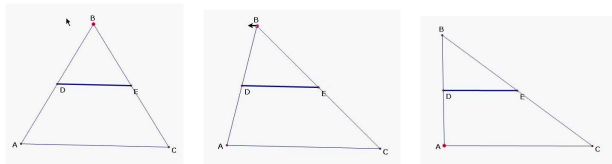
It should be noted that while Barbara explains that she drags point B downward in order to gain a convenient way to observe the figure, Beth suggests that Barbara's strategy also shows that DE and AC actually overlap when they are dragged on top of each other (Line 32). Beth's explanation implies that she assumes that DE and AC would overlap if, and only if, they are parallel to each other. This reaction is similar to the abduction by collapsed figure strategy adopted in Excerpt 8.2 by Alex and Alan, though Beth uses this as a supporting observation rather than an explicit strategy. Beth's reference to the overlapping behaviour may suggest that the distance between DE and AC remains zero throughout when the figure is dragged so that DE is placed over AC.

Both students' responses show that they are aware that in the DGS environment, the tacit property of the figure remains invariant when the figure's shape is changed by the drag-mode. They can then use this dynamic feature to manipulate the figure in such a way so as to demonstrate the property clearly, though this approach remains at inductive reasoning level.

EXCERPT 8.8: Reasoning from a Familiar Territory

When Barbara and Beth discover that the segment connecting midpoints of two sides of a triangle are always parallel to the third side, and are challenged to explain why, Barbara drags the top vertex of the triangle to the left and stops when the triangle looks like a right angle triangle which reminds her of the property of Pythagorean Theorem. Barbara starts to reason from that particular case, referring to the similar property of the right angle triangles with constant length ratio. The following extract shows how Barbara tries to reason from the known geometric property.

Table 8.8 Reasoning from a familiar territory

Dialogues	Actions	Reasoning/Comments	Screen
Barbara: Is the triangle first given as an equilateral triangle, or not? Researcher: No, just a common triangle. Beth: Now it is an equilateral triangle.		Barbara shows her attention to the type of triangle when she asks about the equilateral triangle.	 (a)
	Barbara drags point B to the right (figure c) until it looks like a right angle triangle (figure d)		 (b) → (c) → (d)
Barbara: Is it like the ratio of triangle 3:4:5? Beth: Relevant? Barbara: Like the sides are not equal to each other but the hypotenuse will be 5, the base will be 4 and this (side AB of figure c) will be 3 Beth: Pythagoras? Barbara: Yeah, sort of. And if we halve it, it	Barbara uses the arrow tool to point at the side AB.	Barbara abductively relates the case to the property she learned before about the ratio of similar right angle triangles which Beth recognises from the Pythagorean theorem	

Line	Dialogues	Actions	Reasoning/Comments	Screen
20	will be parallel. With midpoints of the hypotenuse and the cathetus, when we connect them the line will be parallel to the base. I've heard before, the rule about Pythagoras. Beth: But how can we tell that they are 3, 4, 5?		lesson.	

Barbara first directs her attention to the type of triangle the figure can be adjusted to when she asks about the original appearance of the triangle: whether it was an equilateral triangle (Lines 1-2). This may lead her to consider the property of a particular type of triangle she had learned before. She ends up relating it to the constant ratio of the lengths of three sides of a right angle triangle which can be 3:4:5 (Line 10). This reflects the inter-relationship 2-5-6 of the analysis model where students' prior knowledge of geometry plays a role in their reasoning process towards deduction, which is one of the learning objectives of the task.

The flexibility in DGS from the drag feature may show Barbara that the constructed triangle in the DGS environment can be manipulated into any type she wants. She then takes the opportunity to adjust the figure into a right angle triangle and starts to reason from there. The fact that she points out the relationship between the constant ratios of 3:4:5 for a particular shape of right angle triangle suggests the awareness of the property of similar triangles she learned in the past for the right angle triangle case. She may also see that if she can claim that the ratio for the three sides of both triangles is 3:4:5 and the two triangles are both right angle triangles, DE would be parallel to AC since they are both perpendicular to the same segment which is AB. This reasoning involves an inference from a particular case in order to relate it to the known property. It can be interpreted as abductive reasoning where the property of constant ratio of a right angle triangle is linked to the observed figure but the actual relationship may not yet be found. Barbara's belief that the explanation from the right angle triangle case may be later extended to a general case, shows that she believes a right angle triangle to be one instance of a class of 'any' triangle which she later adopts as a different strategy when challenged to explain the case to non-right angle triangle case.

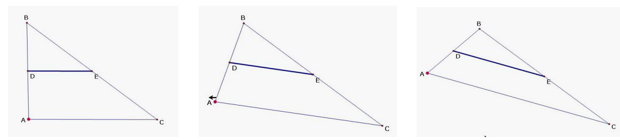
This shows the inter-relationship 2-3-5 where the flexibility of the dynamic feature in DGS helps students to manipulate the figure in a way that helps them reason abductively or deductively, for example by adjusting the triangle into a right angle triangle. This excerpt presents a unique case where the DGS feature directly influences the students' progress in reasoning strategies, toward the stronger claim of deduction from the known rules. Nevertheless, students need to also generalise the case for other types of triangle in order to validate the statement. However, starting from a right angle triangle can provide a helpful stepping stone for the student to spot the relevant property, i.e. the constant ratio of the triangle's sides and the concept of similar triangles. The dynamic flexibility in DGS, where a figure can be manipulated the way the user wants can, therefore, be helpful for their reasoning strategies toward deductive reasoning as well as inductive reasoning as discussed in the previous excerpt.

Nevertheless, Beth finally argues that they cannot be certain that the length of the triangles' sides are actually 3, 4 and 5 units (Lines 24-25) which Barbara cannot figure out. This pair of students then discards the idea to adopt the Pythagorean Theorem to help them explain the parallel property. This limitation reflects the influence of the way the task is designed by the researcher in the first instance, especially when all measurements are excluded. Barbara's proposed strategy to use Pythagorean Theorem by using a certain set of length ratio therefore cannot take place in this research. Such obstruction shows the inter-relationship 2-5-6 in the analysis model where the pre-set scope of the tasks has already framed the possible strategies students may adopt for their justification process.

EXCERPT 8.9: Transformational Reasoning

After being stuck with the attempt at Pythagorean Theorem, the researcher challenges Barbara to explain the case she claims from a right angle triangle to general cases. Barbara then drags the vertex A to change the shape of the triangle into a non-right angle triangle shape. She then switches to a different kind of reasoning in order to explain the similar relationships based on what she mentioned in the previous excerpt.

Table 8.9 Transformational reasoning

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Researcher: What if it is not a right angle triangle?	Barbara drags point A upward (figures b-c) to change the triangle's appearance.		 (a) → (b) → (c)
	Beth: They are still parallel. Researcher: Why? Barbara: It's like the power of amplification or line extension of these two sides will be equal. So it will keep the characteristic of being parallel to each other.		Barbara uses transformational reasoning of amplification process to explain.	

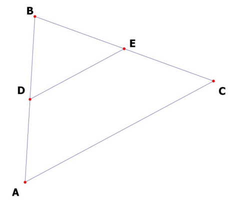
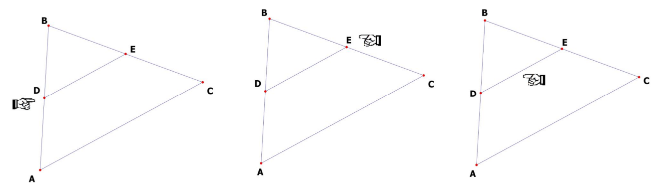
When Barbara is challenged to reason in the non-right angle triangle case, she continues to adopt the concept of constant ratio from the previous excerpt. She further explains that when the figure's shape is changed, the amplification of the line extension of sides AB and BC remains equal (Lines 9-11) which suggests that the ratio remains constant. Segments DE and AC therefore remain parallel. Her explanation clearly involves the characteristic of the change of the figure where sizes are modified but controlled by constant ratio. Such ratio maintenance then keeps the segments DE and AC parallel to each other. However, since Barbara does not elaborate on why the amplification remains constant, this reasoning remains in an abductive level of reasoning, where she simply states the possible reason without providing the logical link. Barbara's reference to transformational reasoning may gain influence from her experience with the dynamic environment in the GSP where the figures can be adjusted dynamically but with certain control or mechanism underlying it.

This is similar to the case of Alex and Alan presented in Excerpt 8.5, where the concept of dilation is used to explain the observed property. Again, this situation reflects the tension in the inter-relationship 1-3-5 in the analysis model where the exclusion from the task of Transformation features in the GSP may prevent the students from using it in their explanation. This recurrence of reference to Transformational geometry in the Euclidean geometry explanation strengthens the fact that students may adopt one idea from a different domain of geometry to explain another. Various commands from different domains of geometry in GSP may therefore enrich the students' strategies in their reasoning process.

EXCERPT 8.10: Deduction from the Construction Process

At one point, when both students are asked to reason why they observe the figure to behave in such a way, Beth turns back to explain the reason by stating the way she and Barbara constructed the figure. Though this reasoning gives no new information for the justification process, it still shows what Beth has in mind when she is considering this problem. The following excerpt shows her response.

Table 8.10 Deduction from the construction process

Line	Dialogues	Activities	Reasoning/Comments	Screen
	Researcher: So why DE remains parallel to AC when the figure's appearance is changed?			 (a)
		Beth adjusts the triangle until it appears as shown in figure a.		
	Beth: It's at the right place. Because this (point D) is the midpoint of this line (AB-figure a) and E is the midpoint of this line (BC-figure c). If we draw a line connect these (points D and E), we'll get another triangle (figure d). So they are parallel.	Beth uses her finger to point at each element while explaining without moving it further.	Beth deductively refers to the construction process to reason.	 (b) → (c) → (d)

Beth's justification strategy in this excerpt is to simply repeat the construction process she and Barbara carried out on the figure. Though this way of reasoning does not provide any further inference, it can imply that Beth believes that the fact that DE is parallel to AC is a result of the fact that they construct D and E as midpoints of the sides AB and BC respectively. This can be deemed a deductive reasoning based on the witnessed development of the figure in the GSP environment.

Beth's answer may also suggest that if D and E are not midpoints of AB and BE, DE may not be parallel to AC. Nevertheless, she does not use modus tollens to demonstrate the statement conversely as in Alex and Alan's case. But her remark that this should be the reason shows that she believes this fact to be an important factor.

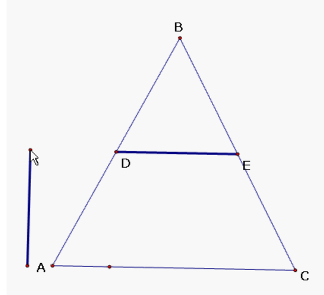
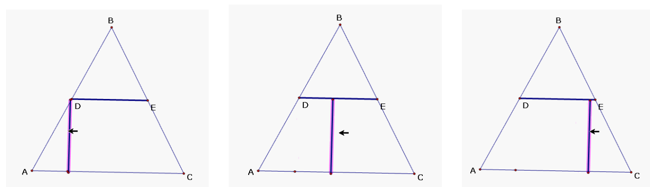
The design of the task to involve the students in the construction process, instead of working from the pre-constructed, plays an important role in declaring the causality of the figure's behaviour, i.e. the parallel property is a result of midpoint constructions in this case. Had the students not known of the construction process, it is possible that they may think that DE is constructed to be parallel to AC. Beth's description of the construction process she uses as justification in this excerpt therefore points out the essential causality of the figure's behaviour, though she cannot yet find the logical link between the process of construction and the figure's behaviour. This reaction reflects the inter-relationship 2-3-5 where the way the task is designed shows how construction in the DGS environment works, which students need to appreciate in order to make sense of the construction's behaviour.

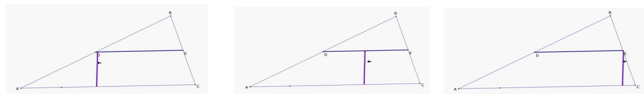
EXCERPT 8.11: Checking with a Construction

After several attempts to explain why segment DE is always parallel to AC with different strategies, Barbara ends up using the simple construction of a new segment to visually demonstrate that the distance between DE and AC remains the same throughout the figure to show that they are parallel to each other. The following excerpt illustrates Barbara's adoption of this strategy.

Table 8.11 Checking with a construction

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Researcher: As you say that when we construct midpoints D and E and draw a segment connecting them, the segment will be parallel to the base. Please discuss to find the reason why. You may construct any additional element or do anything with the figure.			
		(Researcher leaves)		
	Barbara: If we construct another line between DE and AC and compare to see if this line is equal to the distance between them.			
	Beth: How? Let's try.			
		Barbara constructs a new vertical segment then tries to adjust its length so that it is as long as the		

Line	Dialogues	Actions	Reasoning/Comments	Screen
		distance between DE and		
		AC as shown in figure a		
	Barbara: Is it there? Hold on. There?	Barbara tries to make the		
	Beth: Yeah, it's there.	segment's length precise.		
	Barbara: Let's try to move. Like this.	Barbara uses the arrow	Barbara inductively	
	Beth: Ah ha.	tool to drag the segment	demonstrates that the	(a)
	Barbara: And like this.	to the right (figures b, c	distance between DE and	
		then d) in order to check	AC is constant all	
		the distance between DE	through.	
		and AC. Then she moves		
		the segment back and		
		forth.		
	Beth: (laughs) Measure easily like this?		Beth questions the	(b) → (c) → (d)
	Barbara: But it works. See they are		strategy's simplicity.	
	equal!			
	Beth: Is it too easy?			
	Barbara: Is it correct? The distances are			

Line	Dialogues	Actions	Reasoning/Comments	Screen
	the same.			
	Beth: What if it is like this?	Beth adjusts the triangle's appearance as shown in figure e.	Beth challenges Barbara to generalise her strategy with a new case.	 (e)
	Barbara: So we must make the line longer.			
		Barbara adjusts the size of the segment and drags it rightward (figures f, g then h) to check again. (Researcher returns)	Barbara inductively demonstrates the new case.	 (f) → (g) → (h)
	Researcher: So what did you do?			
	Beth: Check it.			
	Researcher: Are they parallel?			
	Both: Parallel.			
	Researcher: Why?			
	Barbara: It's the distance. The distance remains the same.			

From this excerpt, Barbara utilises the segment construction feature of the GSP to generate a segment together with the dynamic feature of the drag-mode, where a segment can be translated across the screen without changing its size (as long as its endpoint is not moved) to demonstrate the parallel property (Lines 23-30). Similar to the case where Barbara uses the drag-mode to help her visually observe that DE and AC are parallel to each other in Excerpt 8.7, she also deduces the property of the parallel lines where the distance between the lines remain constant as her criterion for the inductive observation. However, in this excerpt, Barbara goes a step further in utilising additional constructed objects in the GSP environment as checking tools to see whether DE is actually parallel to AC. The dynamic flexibility in translating the constructed segment around the screen also allows Barbara to continuously demonstrate that the distance between DE and AC never changes and remains equal to the segment's length. It also helps Barbara to demonstrate a thorough case where the constructed segment can be swept throughout the length of DE. Note that Barbara's creative application of the construction feature in the GSP for her justification processes may not conform to the GSP designer's intention of the command. Nevertheless, this strategy reflects the students' unique utilisation scheme of the observed object's behaviour in the GSP environment to help them verify the claim. Features in the GSP can therefore be used by the students in a variety of ways and may not necessarily follow their intended purposes. It can also be analysed that Barbara's adoption of the construction feature as a checking tool is a result of the exclusion of measurement activities in the designed tasks. Students, therefore, need to find a way round this in order to perform the measurement, especially when they feel that measuring or checking is a convincing way of verifying a geometric statement.

Later in the excerpt, Barbara also shows that this strategy is applicable to any triangle when Beth challenges her to try the strategy with a different shape. Barbara manages to argue that the length of the checking segment can be changed but it would remain equal to the constant distance between DE and AC (Lines 37-46). Note that Beth expresses her unfavourable impression of this method by deeming it ‘too easy’ (Line 34). From the Critical Incident Recall interview, Beth further elaborates her reaction to this strategy by expressing that she feels the strategy is too straightforward and not mathematically sound. It does not seem to be the solution to the given problem. This response shows another dimension of the criteria students use to assess the strategy for the task. Beside the validity of the explanation, the richness of mathematical foundation is another essential criterion that would make the strategy acceptable. This illustrates the affective value students place on their justification process where validity and logic are not the only factors for assessing the reasoning strategy.

ANALYSIS OF BARBARA AND BETH’S RESPONSES TO TASK 3.2

Similar to the case of Alex and Alan, Barbara and Beth’s responses to Task 3.2 is analysed by the analysis model presented in Figure 8.1, with the inter-relationships 1-3-5 and 2-3-5 as the focus for this section of data analysis. It is evident that the approach of the designed task and the role of the researcher as a challenger initiate a variety of reasoning strategies from Barbara and Beth. This again reinforces the relationship 2-5 in the analysis model where students are challenged to reason in order to verify the discovered geometric property.

Though Barbara and Beth's collaboration is much more harmonious than Alex and Alan's, there are also some instances where Beth challenges Barbara's strategies, such as when she argues about the practicality of the Pythagorean Theorem-related application, the applicability of the segment as a checking tool to different triangles, or when she questions such strategy's simplicity. These reactions show that setting students to work together in pairs helps enrich the argumentation process among the students beyond the challenge set by the task and the researcher, as shown in the inter-relationship 2-5-6 in the analysis model. The pair workings also help the researcher to elicit the students' thinking process much better than in the individual interview where students' reasoning can be conveniently spotted from their dialogue.

Regarding the use of the GSP to help them reason and to learn Euclidean geometry which reflect the inter-relationships 2-3-5 and 1-3-5 in the analysis model, Barbara and Beth adopt an interesting basic strategy in order to verify the parallel property of segments DE and AC. Instead of showing confidence when they first observe visually that the segment DE is parallel to AC, Barbara utilises features in the GSP to assist her observation before confidently claiming that the segments are truly parallel to each other. She first uses the drag-mode to drag the figure so that DE and AC become very close to each other. This facilitates her observation that the distances between segments DE and AC at the left and the right sides of the triangle are equal. After that she utilises the Segment construction tool to construct a segment as a checking tool to demonstrate that the distances between DE and AC are equal throughout the figure. Barbara's basic strategy is, therefore, to demonstrate that the distance between DE and AC never changes the particular appearance of the figure resulting in the parallel property. This clearly shows her deduction of the known rule about parallel lines in order to validate the inductive observation, which is different from a straightforward claim that 'they look parallel'. Barbara's justification strategy, therefore, incorporates both inductive and deductive parts of reasoning. The parallel lines' property of the constant distance between the lines is used as a warrant. Then the concrete data is sought after using the GSP features to validate such warrant. Using Toulmin's model of argumentation to illustrate this strategy, Barbara's main justification approach can be outlined as shown in Figure 8.3.

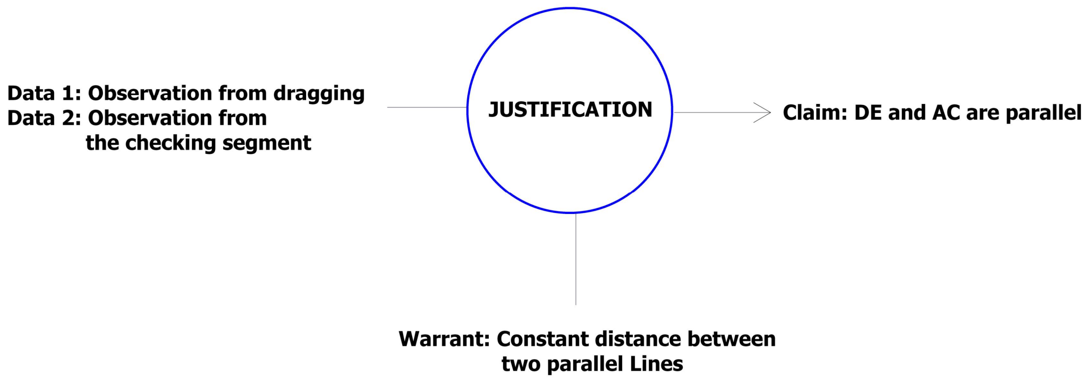


Figure 8.3 Toulmin's model of Barbara's reasoning strategy

Apart from using the GSP features as a facilitating tool for visual observation, the drag-mode in the GSP also gives the students flexibility to manipulate the triangle in a way that helps them reason in the justification process. The fact that Barbara uses the drag-mode to adjust the triangle from an acute triangle to an obtuse triangle in the Excerpt 8.7 in order to explore the parallel property, and to adjust the triangle into a right angle triangle in order to abductively relate it to the lesson about Pythagorean Theorem in Excerpt 8.8, shows her appreciation of the dynamic flexibility in the DGS environment. This flexibility allows her to analyse the figure from a particular case before applying it to the general case. Her adoption of the nearly collapsed figure in Excerpt 8.7, and the right angle triangle in Excerpt 8.8, as particular examples, implies the awareness of the possibility of using the same principle in order to explain the triangle in different shapes. This becomes apparent when Barbara manages to apply the checking segment strategy with a different shape of triangle given by Beth presented in Excerpt 8.11. The dynamic feature in the GSP environment therefore provides the students with a generalised concept of a geometric figure which can be adjusted to look like a particular shape while preserving the fundamental properties. Such flexibility allows the students to utilise a wider range of prior knowledge, such as the constant ratio of the similar right angle triangles in this case to help them in the reasoning process.

As in Alex and Alan's case, Barbara also briefly adopts transformational reasoning strategy to help her explain the observed parallel property when she mentions about the process of amplification in Excerpt 8.9. Though she does not manage to elaborate the further reason for such 'amplification', this reaction suggests that Barbara views the static figure in the GSP as a product of a dynamic process of dilation, which can be as a result of her experience with the dynamic environment. Again, the limitation of the task to Euclidean geometry reasoning without the introduction to Transformation commands may prevent this pair of girls from further exploring this idea. This reflects the tension in the inter-relationship 1-3-5 of the analysis model where the researcher's identification of commands in the GSP to be used as a learning tool frames the reasoning possibility of the students.

Apart from reasoning using the GSP features, Beth also adopts a different approach of deductive reasoning by referring to the process of the midpoint construction as a key factor that makes DE parallel to AC. This reasoning, although it provides no additional information other than the instruction process of the task itself to explain the observed property, still reflects a step in Beth's reasoning process by identifying the relevant known property from the construction. It highlights the importance of letting the students take part in the construction process in the task design which is supposed to be used as a basis for students' reasoning. This relationship is presented in the inter-relationship 2-3-5 in the analysis model where the approach of the task design may directly influence the students' reasoning process.

Since the application of the constant ratio property of the similar triangles to explain the figure's parallel property is not fully developed by this pair of students, even though they briefly touch upon the idea of constant ratio when they mention a particular ratio relating to a right angle triangle in Excerpt 8.8, and the concept of amplification in Excerpt 8.9, the reasoning strategies of Barbara and Beth in this task remain at inductive and abductive reasoning. This is based on visual observation, with some hints of deductive reasoning, when they refer to known geometric property such as the parallel lines property to assist their observation. Nevertheless, their reasoning strategies lead to creative uses of the GSP features to help their reasoning which may not conform to the original purpose of such features. The pair's responses to this task, therefore enrich the application of the GSP as a learning tool to help them reason, highlighting the possible variety in the inter-relationship 1-3-5 of the analysis model.

8.2.3 Conclusions from the Two Pairs' Responses

From the discussions of Alex and Alan together with Barbara and Beth in the reasoning strategies presented above, it can be seen that students are limited when using the DGS features that are introduced in Task 1, and pre-chosen by the researcher. It appears that other commands in the menu excluded from the tasks, such as Transformation can actually be helpful for the students in verifying Euclidean geometry statements. This illustrates the tension between inter-relationship 1-3-5 where the researcher's scope of the commands to be used in the task can limit the student's utilisation scheme in their reasoning process. Nevertheless, both pairs of students adopt transformational reasoning to help them explain the parallel property which can be interpreted as an influence of working in the dynamic environment of the DGS.

Different approaches of reasoning are used to explain the observed parallel property in this task. This is partly as a result of the challenges placed by the researcher on the students to elicit stronger reasoning from them. Nevertheless, the reasoning challenge may also come from the students themselves since the tasks are designed for pairs. This enriches the inter-relationship 2-3-5 where the reasoning challenge is not solely from the task or the researcher but can be from the students themselves.

A range of information is drawn as evidence for students' reasoning, some of it involves the GSP features and some of it does not. For those involving the GSP features, the most common aspects the students use are the dynamic nature of dragging, where they can observe the invariant property when the figure is dragged, the drag-test to ascertain the property and the construction features which students treat as a reliable entity for checking certain property. This confirms the inter-relationship 2-3-5 and 3-5-6 where students actually use features in DGS to learn Euclidean geometry, and realise that using DGS is part of the learning objective. Most of the reasoning strategies that involve DGS features are inductive reasoning. For abductive and deductive reasoning students usually adopt their prior knowledge of geometry or real-life experience to verify without using the dynamic feature in DGS.

The data analysis presented in this chapter illustrates that the four entities identified in the research models are independent and can affect each other. The assumed relationships and inter-relationships are confirmed in several cases showing students' adoption of the DGS feature to support their reasoning toward deduction, which is the objective of the tasks. Tensions between these entities are also present especially when a particular entity limits the possibility of, or contradicts with, another entity. Working with this model also provides further enrichment of data analysis especially when certain part, not originally in the model, plays an essential role such as reasoning without the DGS feature and reasoning challenge in the pairs of students.

The analyses of the rationale for Task 3.2 design, together with the two pairs of students' responses to this task presented in this chapter should illustrate the presence of the assumed inter-relationships identified in the analysis model developed in Chapter 6 for the data analysis phase. The analysis model helps the researcher to realise the actual effects of pertinent entities which can play a significant role in the students' reasoning process in the DGS environment in the light of the data. This helps to provide answers to the main research question regarding the relationship between the DGS environment and students' higher-order thinking of reasoning. The similar approach of data analysis is then applied to the rest of the interview tasks,, as well as the rest of participant students' responses, in order to find answers to the research sub-questions 1 and 2. The findings from the analyses are presented in Chapter 9 and 10 before a thorough discussion of the research result in Chapter 11.

9 LEARNERS' INTERPRETATIONS OF THE DGS

This chapter concentrates on the data analysis of the learners' interpretations of the DGS environment, or the GSP environment in this research. The same analysis approach adopted in Chapter 8 is applied to all nine pairs of students' responses to Task 1. These reactions involve the inter-relationship 3-4-5 in the analysis model as shown in Figure 9.1, and is aimed at providing answers to the research sub-question 1: "How do learners use their geometry knowledge to reason about the way DGS portrays Euclidean geometry?".

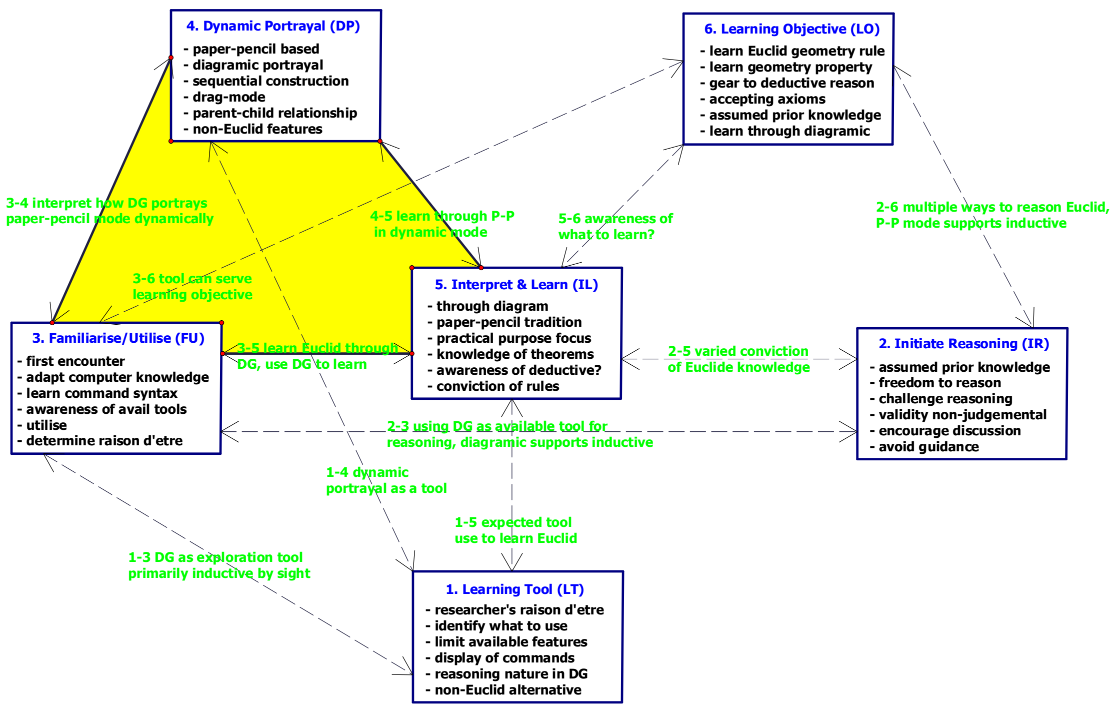


Figure 9.1 Inter-relationship 3-4-5 in the analysis model

The analysis in this chapter highlights the tension between students' prior experience with the paper-and-pencil environment and their first encounter with the DGS. From the data gained in the task-based interviews, the researcher tries to elicit significant themes relating to the unique ways students interpret or perceive Euclidean geometry in the DGS environment, especially those that differ from the paper-and-pencil mode. Students' interpretations of the DGS environment are categorised into two main groups: students' interpretations relating to the non-dynamic portrayal of Euclidean geometry in the DGS, and interpretations relating to the dynamic portrayal of Euclidean geometry in DGS. This is to distinguish relationships 3.Familiarise/Utilise (FU) and 4.Dynamic Portrayal (DP) in the model of study. Where there are several examples of similar cases across these nine pairs, the most illustrative example is chosen to be presented in this Chapter.

9.1 INTERPRETATIONS OF THE DGS' PORTRAYAL OF EUCLIDEAN GEOMETRY

This section focuses on the inter-relationship 1-3-5 of the analysis model which also relates to the inter-relationship 3-4-5. It depicts the way students interpret the DGS features that do not directly incorporate the dynamic feature.

9.1.1 The Plane Boundary of the Sketchpad

The fact that the GSP displays the Euclidean geometrical plane as a white rectangle with horizontal and vertical scroll-bars lends itself to different interpretations from the students. The presence of scroll-bars in the software interface suggests that the rectangle boundary is merely a current visible plane of the infinite boundaries, where students can use the mouse to scroll around the page in the desired direction. Nevertheless, some students interpret the rectangle plane as a finite boundary of the objects they draw. The following excerpts show students' different interpretations of this rectangle boundary on screen.

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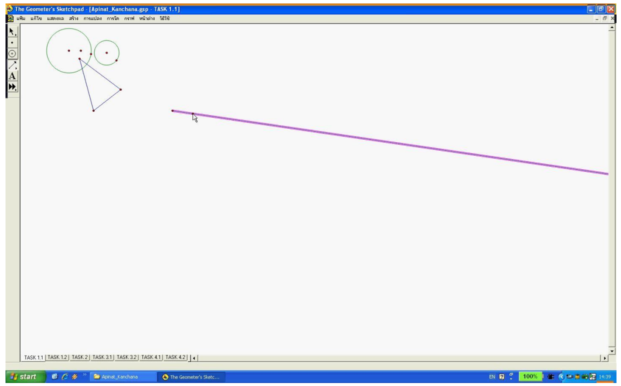
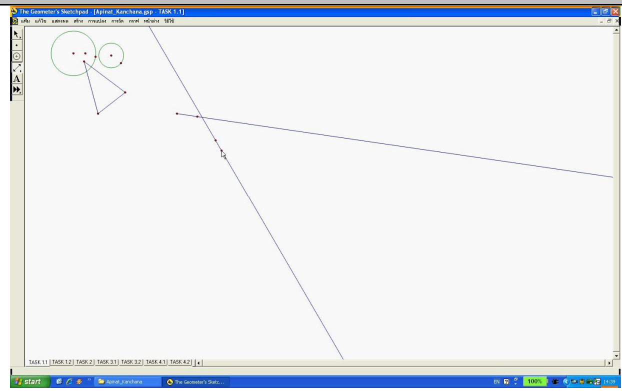
Table 9.1 Charles and Chloe's responses to the GSP boundary.

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Researcher: This command (straightedge tool) has three sub-commands, please try all of them.			 (a)
		Charles first tries the Ray commands and uses the arrow tool to turn it around (figure a).		
	Chloe: Ooh! So long?			
	Charles: Why is it like this? (rather disappointed then giggles shyly)		Charles' question suggests that he does not expect the ray to be this long.	
	Charles: Why is it extended?			
	(3 sec pause)			
	Charles: This one must be long at both sides		Charles abductively predicts the result of the Straight Line command from his experience with the Ray command.	
		Charles moves on to try the straight line command (figure b)		
	Chloe: That's it!			

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Charles: Ahhhhhh! (assured)		Charles appears to be satisfied when he can predict correctly.	 (b)
	Researcher: Why do you think it would be long at both sides?			
	Charles: Because it has two arrows (laughs shyly) so they go very long.	Charles uses his finger to point the command's icon.	Charles' laugh may imply that he finds the reasoning rather straightforward	
	Researcher: Where do they go to?			
	Chloe: The way the arrow points to.			
	Charles: To the edge of the paper			

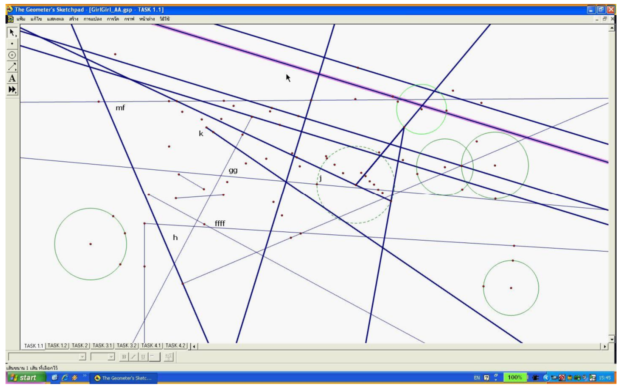
From Table 9.1, Charles and Chloe perceive the ray and the straight line to be a ‘long line’ at one side and both sides respectively (Lines 7 and 12). They do not explicitly refer to the mathematical terms of ‘ray’ or ‘straight line’. Charles manages to adopt abductive reasoning to predict the relationship between the visual arrow head of the command’s icon with the extended length (Line 23). He first learns about this relationship from the trial of the Ray command and assumes that the two arrow heads in the Straight Line command icon would behave similarly which turns out to be correct. This reasoning may be categorised as an abductive reasoning, since Charles does not give any logical explanation or inference to deductively reason why the arrow heads in these commands should result in an infinite property of the line. His abductive reasoning can also be classified as an ‘over-coded abduction’ since he is very certain about this relationship and seems to ignore other possibilities.

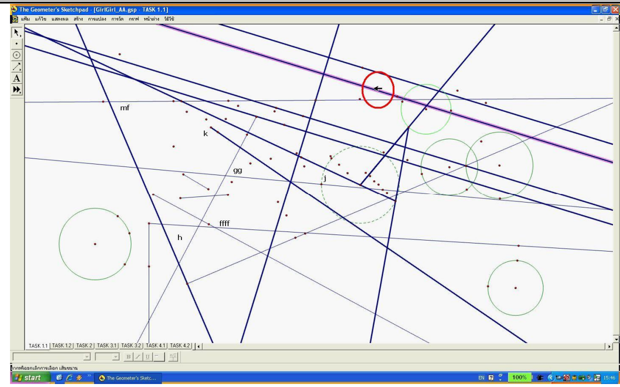
It is obvious from this excerpt that neither Charles nor Chloe recognise or are aware of the three distinct geometric straight objects: a segment, a ray and a straight line. They call the lines either long or extended without an explicit reference to the infiniteness. This may be the reason that Charles believes the GSP’s sketchpad on screen to be a rectangular piece of paper which is clearly expressed when Charles mentions that the line goes to the ‘edge of the paper’ (Line 27). This reaction illustrates that without the students’ perception of the possible infinite property of geometric shapes, the scroll-bars in the GSP provided as a tool to explore the infinite plane, can be meaningless for the students. Students, therefore, relate the GSP screen to the physical sheet of paper they are familiar with in the traditional paper-and-pencil environment. The more theoretically faithful portrayals of the ray and the straight lines in the GSP are not appreciated by the students in this case. They even believe these shapes to be alien and do not realise their benefit.

Charles and Chloe's reaction to the infinite property in the GSP in this excerpt illustrates the tension in the relationship 3-5, where students' prior experience with the paper-and-pencil environment affects the way they appreciate such unique property portrayed in the GSP.

Such unawareness of the infinite boundary portrayed in the GSP can be an issue when students try some particular commands, as can be seen in Alice and Alma's case presented in Table 9.2.

Table 9.2 Alice and Alma's responses to the GSP boundary

Line	Dialogues	Action	Reasoning/Comments	Screen
		Alice and Alma try to use the Midpoint command but cannot figure out what kind of object the command is for. They try various objects with no success before trying to use it with a straight line		
	Alma: The midpoint of what? Alice: I don't know! (discouraged)	Alma clicks to select a straight line but the midpoint command is still disabled (figure a)	Alma and Alice get confused with the function of the midpoint command.	 (a)
	(12-sec pause) Researcher: You try to use it with this line. What do you expect it to do? Alma: What did you say?			

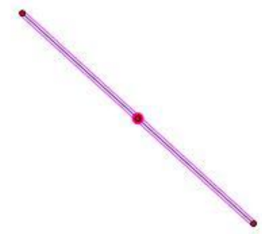
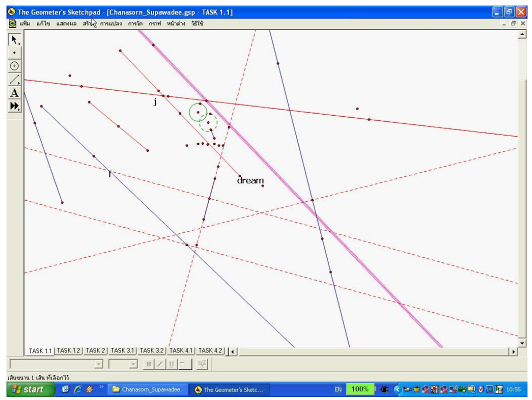
Line	Dialogues	Action	Reasoning/Comments	Screen
	Researcher: You try the midpoint command with this line. What did you expect the software to do? Alma: There should be a point on the line at the middle here. Alice: Right at the middle	Alma uses the arrow tool to point to the position where she thinks the midpoint of the straight line is (see the bold circle in figure b).	Alma and Alice expect a midpoint at the middle of the visible infinite line.	 (b)

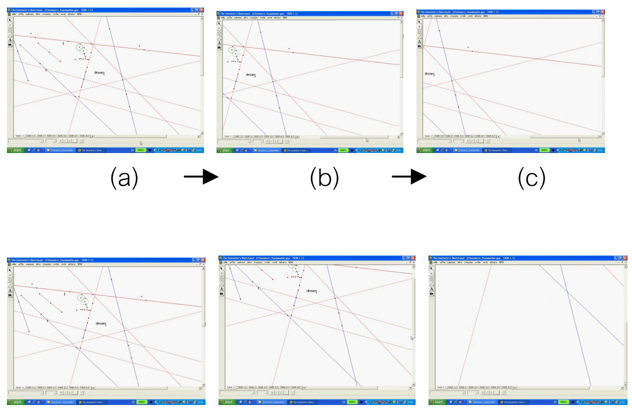
In this excerpt, Alice and Alma have difficulty understanding the Midpoint command in the GSP. According to their responses at Lines 21-23, the girls seem to see the infinite straight line as a segment, with the endpoints lying at the top and right borders of the GSP screen. With the Midpoint command, they therefore expect it to help them construct a point right in the middle of that segment.

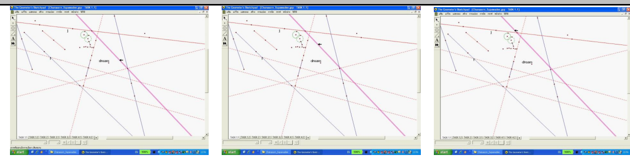
The students' unawareness of the infinite property of the straight line portrayed in the GSP contradicts with the GSP's geometrical treatment of infinite shapes. Such contradiction prevents this pair of students from making sense of the midpoint command since they expect it to work with a straight line, seen as a finite segment. Similar to Charles and Chloe's case, Alice and Alma do not recognise or distinguish the lines in the GSP as a segment, a ray or a straight line. They simply call them a 'line'. They are also unaware of the function of the scroll bars which would help them explore the infinite plan portrayed in the GSP.

Carla and Carol, on the other hand, manage to reason why the Midpoint command does not work with a straight line. They first get confused when the command works with a segment but not a straight line. However, Carla later realises that the midpoint construction command does not work with a straight line because it is too long. This leads her to use the scroll-bar to examine that the straight line is actually extended beyond the visible screen. The following excerpt shows Carla's reasoning together with her experiment with the scroll bar to examine the figure.

Table 9.3 Carla and Carol's responses to the GSP boundary

Line	Dialogues	Actions	Reasoning/Comments	Screen
		Carla tries the Midpoint construction command with a segment (figure a)		
	Carla: Here! Got it!			
5	Researcher: How is it?			
	Carla: We need to use this line (segment). Click at the line and it will show the command.		Carla reasons abductively from a single instance.	(a)
10		Carla then tries with a straight line (figure b).		
	Carla: This one it does not show! Why does it not show?		Carla discovers a different response when she tries the command with a straight line.	
	Researcher: Mmm, why it does not show? (3-sec pause)			
15	Carla: I don't know either (5-sec pause) maybe it is too long.			
	Researcher: How?			(b)

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Carla: Mmm, it's not relevant. Maybe it is not relevant.		Carla may feel reluctant after the researcher's challenge.	
	Researcher: What do you mean by 'too long'?			
	Carla: This line is too long. We don't know where it ends.			
	Researcher: So?			
	Carla: I don't know how to explain.			
	Researcher: What do you think?			
	(10-sec pause)	Carla starts to use the horizontal scroll-bar by moving the bar to the left (figures a-c) when she sees that the line is actually extended. Carla also tries the vertical scroll bar (figures d-f)		 (a) → (b) → (c) (d) → (e) → (f)
	Researcher: Where do you think the			

Line	Dialogues	Actions	Reasoning/Comments	Screen
40	midpoint is? Carla: It should be in the middle. But for this line it doesn't know where the middle is. It's too long.	After trying the scroll-bar, Carla uses the arrow tool to slide the straight line upward several times (figures g-i).	Carla may slide the straight line upward in order to examine that the line has no endpoint.	 (g) → (h) → (i)

From her trials of the Midpoint command with a segment and a straight line, Carla notices different outcomes. The command constructs a midpoint for the segment but remains disabled when a straight line is selected as an active object. In order to make sense of these different software responses, Carla adopts an abductive reasoning by discriminating the segment and a straight line, and conjectures that the two straight objects' different length may be the cause of the Midpoint command's different behaviour. With the idea that the straight line may be too long for the Midpoint command, she then inductively explores the length of the straight line with the scroll-bars and an arrow tool, and discovers that the length of the straight line is extended beyond the visible screen. After the inductive exploration, Carla deduces from her knowledge of what a midpoint is to reason that the Midpoint command does not work with a straight line because its length is too long so the midpoint cannot be located.

Carla's response to the Midpoint command in this excerpt shows a complex situation where the student needs to adopt abductive, inductive and deductive strategies in order to make sense of the software behaviour. Carla does not seem to fully appreciate the infinite property of the straight line at first, but abductively conjectures that the different lengths of the segment and straight line may be the reason (Line 16). However, once she inductively examines that the length of the straight line is infinite by using the scroll-bars (Lines 27-34), she manages to use this data together with her knowledge about the definition of a midpoint to deductively conclude that the midpoint command cannot be used with an infinite straight line (Lines 37-39).

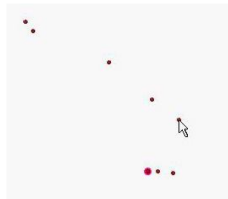
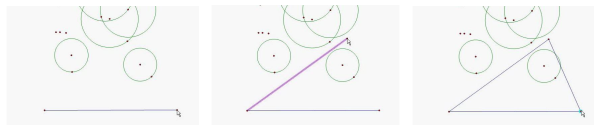
This excerpt shows how the unique portrayal of infinite objects in the GSP helps Carla learn about the infinite property of some geometric objects she possibly never experienced from the traditional paper-and-pencil environment. It clearly reinforces the inter-relationship 3-5-6 where working in the DGS environment refines the students' understanding of a certain geometric property.

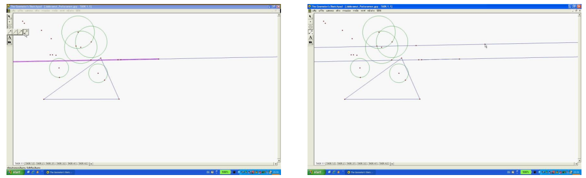
From the three pairs of students' responses to the boundaries of the GSP sketch, it can be seen that students' experience with the paper-and-pencil environment plays a significant role in their appreciation of the infinite plane portrayed in the GSP. Some students believe the visible rectangle screen to be a finite plane; like a piece of paper, while other students manage to use the scroll-bar to navigate the boundless plane beyond the GSP screen and make sense of the infinite property. This unique way the GSP portrays the Euclidean plane provides a good example of the possible tensions between the Learner, DGS and Euclidean Geometry entities in the model of study, especially when the students interpret the Euclidean geometry property differently from the DGS, based on their experience in the paper-and-pencil environment, even though such DGS portrayal aims to respect the object's geometric property more faithfully.

9.1.2 Irrelational Constructions

The fact that the GSP identifies and organises its commands by the most elementary geometric construction such as a point, a circle, a segment or a point on an object, providing the foundation for users to construct a more complicated figure, may cause students' confusion of each command's purpose in the practical sense. This is because some elementary commands in the GSP seem meaningless to them. This notion reflects student opinion on what Euclidean geometry is actually for in real life, especially when most of them believe Euclidean geometry to be a tool for constructing a useful figure, not a body of knowledge in its own right. The following excerpts show students' reactions pertaining to this notion.

Table 9.4 Aaron and Anna's responses to the GSP's elementary commands

Line	Dialogues	Actions	Reasoning/Comments	Screen
		Aaron tries the Point command from the toolbox several times on the plane.		
	Anna: It's a point.			
	Aaron: Point? What is it used for, Anna?		Aaron tries to reason the purpose of point command.	
	Anna: We can draw lines connecting them.			
	Researcher: How?			
	Anna: Connect between the points.			
	Aaron: I'm thinking what it is for. It's a point with colour. What can we use it for?	Aaron keeps trying the Point command.		
	Anna: What is a point with colour? I'm confused.		Anna does not understand what Aaron is trying to figure out.	
	Aaron: Maybe just to make a point!			
	Aaron: This one is to construct a straight line (segment). This can be used to make a mathematical problem.	Aaron then tries the Segment tool where he finally uses it to construct a simple triangle (figures a-	Aaron points out that the tool is practical at least	 (a) → (b) → (c)

Dialogues	Actions	Reasoning/Comments	Screen
Anna: A triangle. Aaron: Math problem like geometrical shapes, 2-dimensional shapes with width and length. Anna: It provides everything for you.	c).	for a teacher to make a problem.	
	Then the students are encouraged to use a Straight Line command which Aaron ends up constructing a parallel line (figures d-e)		 (d) → (e)
Aaron: This one can be used to construct parallel lines. Researcher: How? Aaron: The line command with arrows can be used to construct parallel lines. Anna: It goes straight. Aaron: We can make it precise.		Aaron exclusively claims that a straight line is used to construct a parallel line. He does not recognise that the command can be used for another purpose.	

This excerpt shows that Aaron finds many basic commands in GSP such as Point, Segment or Straight Line meaningless in their own right. He has a tendency to believe that these commands have meaningful and practical purposes, such as for a teacher to design a mathematical problem (Lines 16-17), to construct geometrical shapes (Lines 19-20), or a Straight Line command is used to construct parallel lines. The fact that he claims that a straight line command is used exclusively to construct parallel lines (Lines 32-33) implies that Aaron does not realise that a straight line is useful for other geometrical purposes. He may consider that the infinite length of the straight line is useful in emphasising the definition of parallel lines, where the lines never cross each other and their infiniteness would demonstrate this. This reaction shows that Aaron considers this ‘software’ a ‘tool’ to achieve something practical and meaningful. He does not consider it a model of Euclidean geometry directly. While Anna responds more naturally to how GSP organises the command and accepts this approach of command organisation.

Nevertheless, Aaron’s response in this excerpt clearly illustrates the student’s justification of commands in the GSP, where the function as well as the purpose of each command needs to be clarified in order to learn them. This justification also calls for a reasoning process from the students in order to make sense of the command. Most students, therefore, need to deduce their Euclidean geometry knowledge, or experience gained from past geometry lessons, in order to justify the commands’ presence in the GSP, as exemplified by Aaron’s reaction in this excerpt. Aaron’s uncomfortable reaction towards the Point construction command, when he cannot figure its practical purpose, depicts how the command justification process is needed to support students’ familiarisation of the software. This process of command justification involves the inter-relationship

1-3-5 in the analysis model, where the learner needs to understand how the DGS as a learning tool works, in order to help them learn Euclidean geometry through the tasks.

9.1.3 Objects' Dependencies

In order to control the movement of objects under drag-mode, DGS incorporates the parent-and-child relationships between the constructed objects where a certain object would depend on another based on the order of construction. Such a relationship controls not just the way objects are moved in relation to each other, but also when a certain object is deleted. The following excerpt shows the students' response and interpretation of different behaviour when a circle's centre or circumference is deleted.

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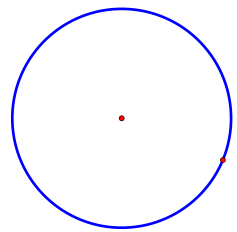
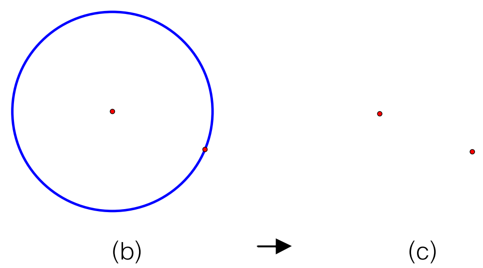
Table 9.5 Brian and Bruce's reactions to GSP objects' dependency

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Researcher: Please construct a circle here.	Bruce uses the compass tool to construct a circle (figure a).		 (a)
	Researcher: Please delete just the circle's centre.	Bruce uses an arrow tool to select the circle's centre then presses the DEL button.		
	Bruce: The whole circle's gone! (surprising)		Bruce' surprising tone indicates that he does not anticipate this behaviour.	
	Researcher: Why is the whole circle gone?			
	Bruce: We clicked the centre then pressed the DEL button. Maybe the command covers the whole circle.		Bruce's reference to the command's behaviour suggests his abductive conjecture of the software's algorithm.	
	Researcher: How?			

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Brian: It's like this is the centre. If it disappears, the circle would also disappear.		Brian refers to the circle's parts' coexistence as the reason.	
	Bruce: Because it (the circle) is depended on this (the centre).		Bruce explicitly quotes 'depended' as a relationship.	
	Researcher: What if we delete just the circumference?			
	Brian: The centre should also disappear.			
	Bruce: The centre should not disappear!		Bruce argues with Brian.	
		Bruce tries to delete just the circumference (figure b), the centre and the point on the circumference remains (figure c).		
	Bruce: See? Because the centre is not depended on the circumference.		Bruce refers to the dependent relationship again.	

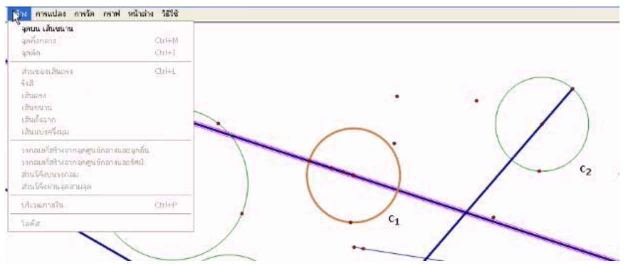
During the GSP commands exploration task, the researcher also asks the students to try to delete a particular part of an object and observe the result in order to examine their reaction to the parent-and-child relationship in the non-dynamic circumstance. According to Brian and Bruce's responses in this excerpt, it is clear that Bruce has an understanding of the dependency of elements in the same figure. He explicitly uses the term 'depended' (Lines 21 & 33) as the relationship which results in the figure's responses to the delete command. This understanding may stem from the fact that, when using the compass tool, they need to mark the centre of a circle first before the circumference can be constructed, allowing them to adjust the size of the circle. This order of construction may help students appreciate the dependency of the constructed objects, where the element which constructed later would depend on the element constructed earlier. Bruce can also correctly predict the result of the contrary case where the only circumference of the centre is deleted, though Brian's briefly asserts the idea of mutual dependency (Line 25).

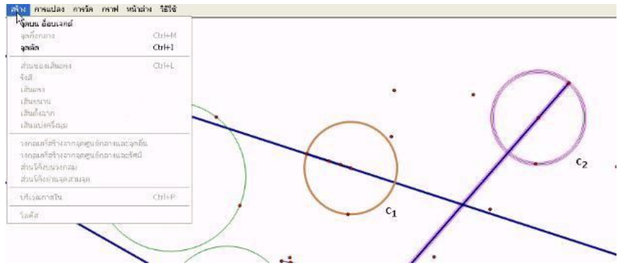
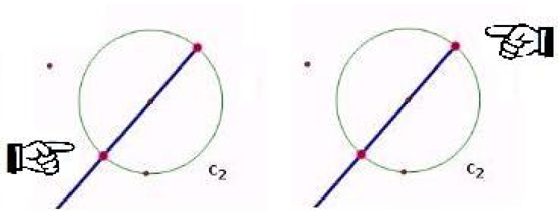
In order to make sense of this GSP behaviour, Bruce also adopts a creative abductive reasoning to conjecture that when the DEL button is applied to the circle's centre, the GSP also applies the command to the whole circle (Lines 12-14). With this abductive conjecture, Bruce then moves on to explicitly claim that the circle depends on the centre, resulting in the chained effect of the DEL command on the centre, while Brian's reasoning at Lines 18-20 and Line 25 implies his perception of the circle's centre and the circumference as the same unit. Without each other, the other element cannot exist. Bruce and Brian, therefore, perceive the circle's elements in the GSP environment differently. While Bruce views the circle's centre and its circumference as separated depending entities, Brian views them as a composite of the same object that must exist together. Nevertheless, the two boys in this excerpt appreciate the concept of the dependency of the geometric object's element in the GSP environment. The only difference is the characteristic of such dependency and whether it is one-way or mutual.

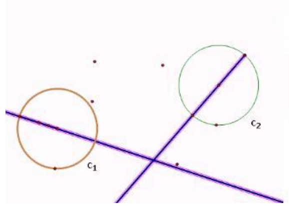
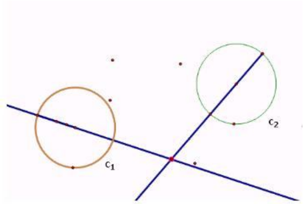
9.1.4 Figuring Out the Construction Commands

Though participant students in the main study can generally make sense of the commands in the toolbox, the Edit and Display menus selected for Task 1, either by abduction to similar commands in other computer software, or by deduction from their prior knowledge of Euclidean geometry, most of them have difficulty figuring out how to execute some Construction commands in the GSP. The fact that GSP's designer adopts the object-before-command algorithm for its command execution appears to cause confusion to students, especially when they have a different understanding or anticipation of how each command should work. The following excerpts from Bob and Bridget's trials with the Intersection and Parallel Line commands illustrate such confusion.

Table 9.6 Bob and Bridget's trial of the Intersection command

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Bob: Mmm. It won't show. Researcher: What do you think the Intersection command should do? Bob: Probably to construct a point on a line. Putting a point on a straight line. Researcher: Can you try?	When the researcher asks Bob and Bridget to try the Intersection command, Bob uses the arrow tool to select objects he and Bridget constructed one by one but observes that the Intersection command is still disabled. Bob selects a straight line but the command is still	Bob's anticipation of the command shows his inaccurate understanding of an intersection point.	 (a)

Line	Dialogues	Actions	Reasoning/Comments	Screen
		disabled (figure a).		
	Bob: It won't show (disappointed).			
	(10-sec pause)			
	Researcher: So when you select a straight line, it won't show. What should be selected then?			
	Bridget: Maybe a line a circle there.	Bridget uses her finger to point at the circle and the line. Bob selects them for her and tries the command again (figure b)	Bridget suggests Bob to select two intersecting objects for the first time.	 (b)
	Bob: Here it comes! (delighted)			
	Researcher: So what happens?			
	Bridget: The intersection points.			
	Researcher: Where?			
	Bridget: Here.	Bridget uses her finger to point at the two intersection points (figures c-d).		
	Bob: Where the straight line passes the circle.		Bob tries to define the intersection point from	(c) → (d)

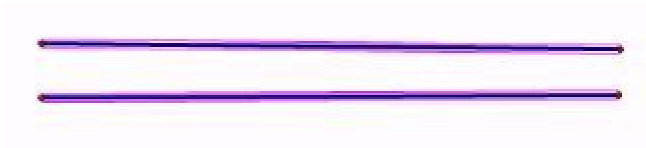
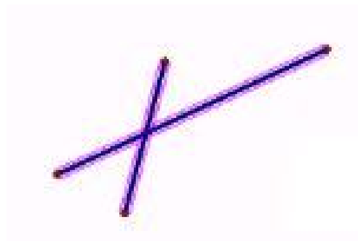
Line	Dialogues	Actions	Reasoning/Comments	Screen
40	Researcher: So what is the Intersection command for? Bob: Mmm, should be point that two lines cross. Researcher: Two lines cross? Can you show it? Bob: Let's try with these. Bob: Yeah, there's a point where the lines cross.	Bob tries the command with two intersecting straight lines (figures e-f).	the command's execution. Bob generalises the command to other objects.	 (e)  (f)

From this excerpt, Bob's inaccurate understanding of the intersection point misleads him to anticipate that the Intersection command in the GSP should help him construct a point on an existing line (Lines 13-14). He tries to select various objects on screen one by one but none makes the Intersection command enabled which confuses him. This reaction shows how students' inadequate knowledge of a particular geometrical concept may prevent them from understanding the GSP commands. However, Bridget later suggests that he should select two intersecting objects, such as a circle and a line, which finally enables the command (Line 24). Bob, therefore, learns the command's function from the output and finally generalises the command feature from a circle and a line to any intersecting objects (Lines 38-39). Bob's realisation of the Intersection commands is, therefore, a result of the contribution of Bridget's knowledge and the confirmation of the GSP output. The excerpt illustrates a case where the student learns a geometric concept of an intersection point by trialling the commands in the GSP environment.

Bob and Bridget find difficulty again when they try the Parallel Line command. Though this time the students have a clear understanding of parallel lines, they have a different idea of how the Parallel Line command in GSP should work, resulting in a conflict. The next extract illustrates how Bob and Bridget get confused with the Parallel Line command.

Table 9.7 Bob and Bridget's trial of the Parallel Line command

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Researcher: Please try the Parallel Line command.			
		Bob selects several objects one by one but the Parallel Line command is still disabled.	Bob adopts the trial-and-error strategy to try the Parallel Line command with various objects.	
	Bob: It won't show.			
	Researcher: What do you expect the command to do?			
	Bob: Don't know.	Bridget just smiles shyly.		
	(5-sec pause)			
	Researcher: So what are Parallel Lines?			
	Bob: Lines that won't meet. They just go straight ahead.		Bob gives a definition of parallel lines based on their property.	
	Researcher: Mm, so what is the Parallel Line command for?			
	Bob: Mmm, maybe we need to construct		Bob picks a strategy	

Line	Dialogues	Actions	Reasoning/Comments	Screen
	two lines. Bob: It won't show. Researcher: Why won't it show? Bob: Maybe they should be like this. Bob: No, won't show. Researcher: Why did you draw the segments this way? Bob: Just tried, when I put them parallel. It doesn't work. So I put them this way. Researcher: What do you expect the	Bob uses the Segment tool to construct two almost parallel segments (figure a) and tries the command. Bob then constructs another two intersecting segments (figure b) and tries the command again.	based on his definition. Bob adopts the trial-and-error strategy with different segment	 (a)  (b)

Line	Dialogues	Actions	Reasoning/Comments	Screen
	command to do with these segments? Bridget: It should adjust these two segments to be parallel to each other.		orientation. Bridget explains her expectation.	

From this excerpt, Bob and Bridget do not realise that they need to select a straight object and a point in order to enable the Parallel Line command. The command would then construct a new line parallel to the original line passing through the selected point. The students instead anticipate that the Parallel Line command should help them adjust the non-parallel straight objects to become parallel to each other. Bob's description of the parallel lines at Lines 13-14 shows that he understands its key concept. Nevertheless, he and Bridget cannot figure out how the GSP manages such property. The object-before-command algorithm in the GSP can therefore confuse the students even though they have a clear concept of what the end product should look like. This reflects Hardy and Wilson's critique on the GSP design (1995) discussed in Chapter 2. The similar reaction and interpretation of Parallel Line command in the GSP is also found by participant students during the pilot study presented in Chapter 7, as well as other pairs of students in the main study. The trial-and-error approach becomes students' common strategy when they find that the command does not work with the objects that they think it should.

9.1.5 Students' Application of GSP features

After the students become familiar with the selected commands in the GSP, the researcher invites them to use the commands they learnt to construct any picture they like in order to examine their view of GSP utilisation as a tool. The students' drawings can be categorised into three groups as follows.

a) Random Collections of Geometric Shapes

Some pairs of students use the GSP to draw multiple geometric shapes in an abstract arrangement without a clear depiction of any meaningful objects. Their drawings are simply random sketches that they can use to try a range of commands they learned, such as the Point tool, the Compass tool, the Parallel Line or the Perpendicular Line, without any other clear purpose. Examples of this group of drawings are shown in Figure 9.2 and Figure 9.3. Note that Brian and Bruce also use the Colour command to apply different colours to the sketch.

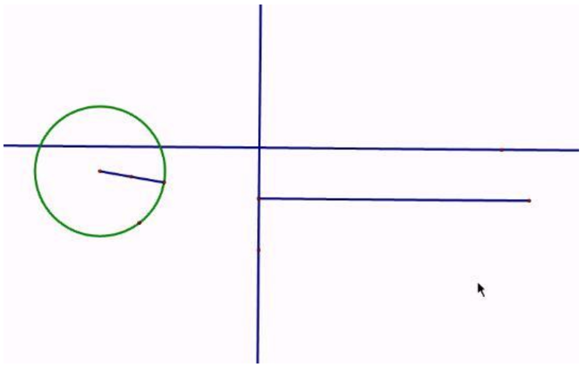


Figure 9.2 Bob and Bridget's drawing

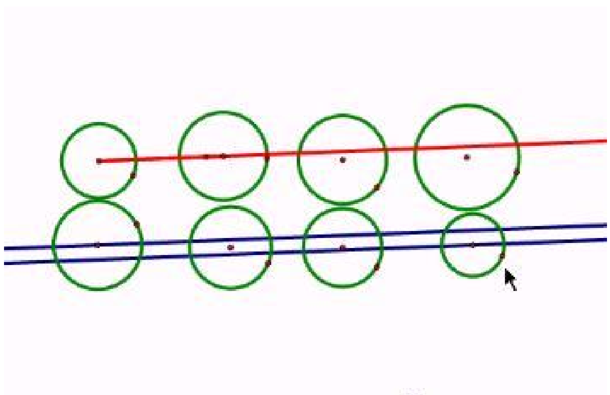


Figure 9.3 Brian and Bruce's drawing

This group of drawings suggest that students view the GSP simply as a tool that provides these geometric construction features.

b) Sketches of Three-Dimensional Shape

Other pairs of students adopt the GSP features to construct meaningful three-dimensional sketches. Carla and Carol exclusively use the Segment tool to draw an isometric rectangular box as shown in Figure 9.4. Note that they draw the parallel segments by sight without the use of the Parallel Line command.

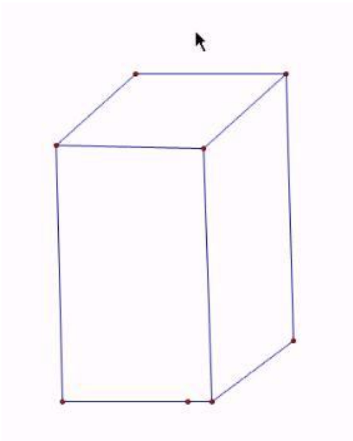


Figure 9.4 Carla and Carol's drawing

Barbara and Beth also use the Segment tool to construct a three-dimensional sketch of a house as shown in Figure 9.5, though they do not manage to present the sketch in an isometric view. Similar to Carla and Carol's drawing Barbara and Beth draw parallel segments by sight without the use of the Parallel Line command. However they utilise the Colour command from the Display menu to make the house more colourful.

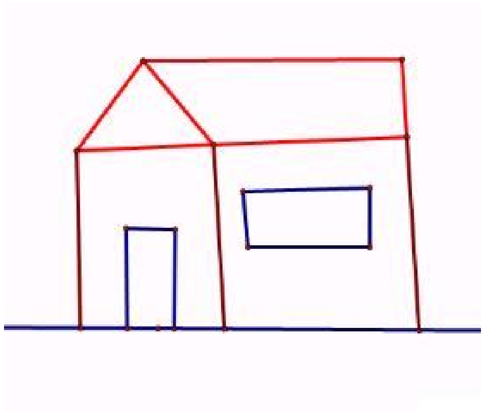


Figure 9.5 Barbara and Beth's drawing

The reason that these students do not use the Parallel Line command for parallel parts may stem from the fact that the GSP produces only an infinite parallel line which would make the drawing look awkward. These two pairs of girls therefore resort to the Segment tool with which they can control the length, and try to draw these simple, recognisable sketches. This group of drawings suggest that students consider the GSP to be a tool to draw a sketch with three-dimensional detail, where the designer can present a three-dimensional shape to communicate to the viewer what the object should look like in real-life. It also shows how students perceive the usefulness of a basic geometric shape such as a rectangle, a parallelogram, a rhombus, or a triangle in order to present these meaningful sketches with an isometric view. This perception shows the application of Euclidean geometry in a real life activity. These Students' view of Euclidean geometry is therefore, not just a set of related theories about shapes but also a useful and practical subject.

c) Drawings of Creative Objects

Certain pairs of students treat the GSP as a drawing tool and use its features to draw two-dimensional creative objects as shown in Figure 9.6 and Figure 9.7.

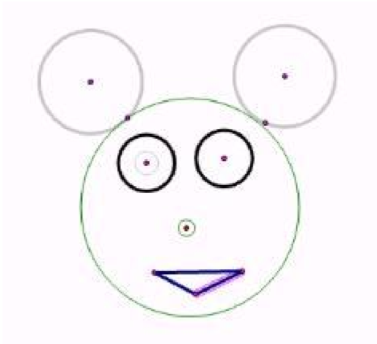


Figure 9.6 Charles and Chloe's drawing

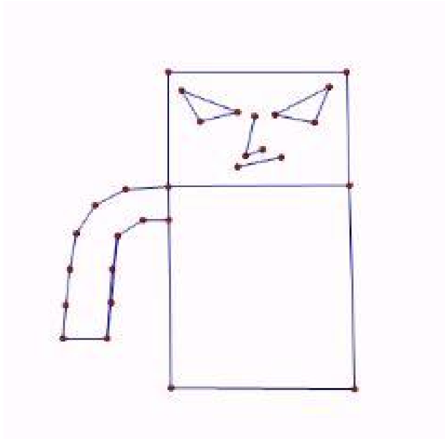


Figure 9.7 Alex and Alan's drawing

These two pairs of students repeatedly use the same commands such as the Compass tool or the Segment tool to help them construct creative pictures of a cartoon face and a robot. They seem to have in mind a vision of what to draw and use the suitable GSP features to create such a vision on screen. Note that Charles even mentions during the task that this software is very much like the Paint programme he used before. This reaction clearly illustrates the students' perception of the GSP tool as a drawing programme without much geometric concern.

These three groups of drawings show students' different views of the usefulness of the GSP's features, ranging from a simple geometric construction tool to a versatile drawing tool. The attractiveness of various features in the display menu, such as colour or line appearances, may lead students to appreciate the GSP as a good tool to draw beautiful pictures. This can be a reason for some students to deliberately avoid the use of commands such as the Parallel Line or the Perpendicular Line, to produce an infinite line, being alien and possibly useless to them. Students' perception of how the GSP can be useful also reflects the students' views of the benefit of Euclidean geometry. Some students tend to try and apply knowledge of shapes in Euclidean geometry to everyday life in order to justify the reason for learning this subject.

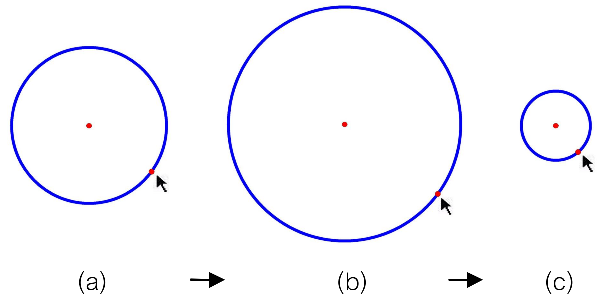
9.2 INTERPRETATION OF THE DGS ENVIRONMENT RELATING TO THE DYNAMIC FEATURE

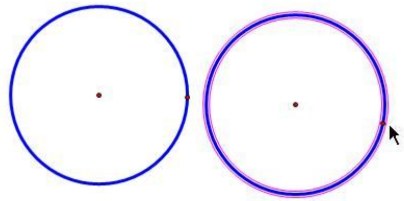
This section focuses on students' interpretations of the dynamic feature incorporated in the GSP environment while working on Task 1 of the designed activities. It reflects the inter-relationship 1-4-5 of the analysis model which is directly related to the inter-relationship 3-4-5.

9.2.1 First Encounter of the Dynamic Feature

Almost all students encounter the dynamic feature of GSP for the first time when they use the Compass tool to construct a circle. They observe that the Compass tool first gives a point on the plane when they click it. Then a circle appears together with another point on the circumference and they can choose to put it anywhere on the sketch. The most common response from students when they observe the flexibility of the dynamic feature is that the software allows the user to 'enlarge' or 'adjust the size' of the object, which implies their perception that GSP can be used as a drawing tool. The following excerpt shows Colin and Conrad's reaction when they try to use the Compass tool.

Table 9.8 Colin and Conrad's reaction to the Compass tool

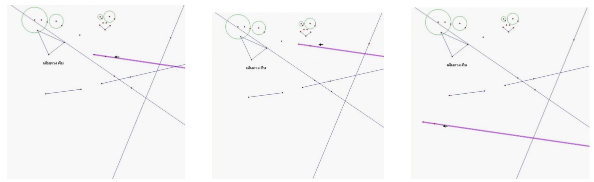
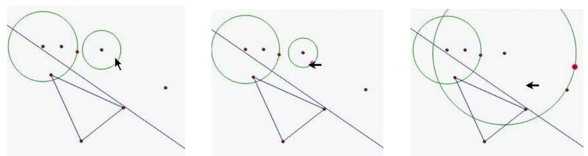
Line	Dialogues	Activities	Reasoning/Comments	Screen
		Colin tries the Compass tool. Once the circle appears (figure a), he moves the mouse to enlarge (figure b) and shrink the circle (figure c).	Colin notices immediately that the constructed circle can change the size in the dynamic environment.	
	Colin: This command enlarges the circle. Researcher: What is it for? Colin: What is it? (5-sec pause)		Nevertheless Colin mistakes the Compass tool as a tool to enlarge the circle.	
	Conrad: Like a compass.		Conrad compares the tool with a physical compass.	
	Researcher: How is it like a compass? (12-sec pause) Researcher: Just say what you think.			
	Conrad: Makes circles of same size.	Conrad uses the Compass	Conrad refers to the use	

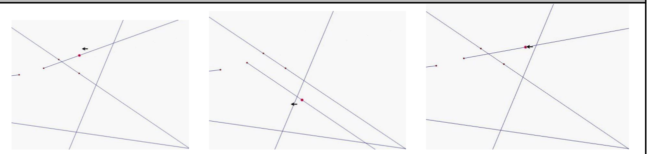
Line	Dialogues	Activities	Reasoning/Comments	Screen
20	Researcher: How do you know that they are equal? Conrad: Estimate.	tool to construct two the circles trying to make them the same size by eye (figure d).	of a compass to construct two circles of the same size when the angle of the compass doesn't change.	 (d)

From this excerpt, Colin appreciates the dynamic feature of GSP immediately when he sees that the Compass tool can construct a circle which he can move to adjust the size as he wants. He first describes the Compass tool as the tool to change the size of the circle, not just constructing a circle (Line 7). Colin's response shows how he identifies the dynamic feature of the Compass tool as the defining property of the command and claims the resizing ability of the command as its main purpose. This is in contrast to Conrad's description of the command by comparing it to the physical compass he used in the paper-and-pencil environment (Line 11), which is also used to produce circles. It should be noted that Conrad and Colin do not realise that the command is also called the 'Compass tool' by the GSP. Conrad uses the term 'compass' himself. Conrad also refers to the special feature of the physical compass to conveniently draw multiple circles of the same size (Line 17) with the constant compass angle. He later adopts the dynamic flexibility of the Compass tool in the GSP to try to draw two circles of the same size by sight to demonstrate the similarity of the Compass tool in the GSP to the physical compass (Lines 17-22). Colin and Conrad's responses in Table 9.8, therefore, illustrate the students' appreciation, as well as the adoption, of the dynamic feature of the Compass tool to construct a circle the size they want, though Conrad may not fully realise such flexibility and still tries to make the command work similar to the physical compass. His justification of the equal circles is, therefore, inductively based on sight. However Conrad's reference to the physical tool shows how his experience in the paper-and-pencil environment influences the way he interprets the command in the GSP.

In the second phase, students encounter the dynamic feature in the GSP when they are encouraged to use the arrow tool with the constructed objects. The students then realise that the arrow tool can be used to select the object (turning it pink), to move or to drag it around as well as to change its size. Charles and Chloe's responses when they try to use the arrow tool depict how they recognise the different purposes of the dynamic feature in the GSP.

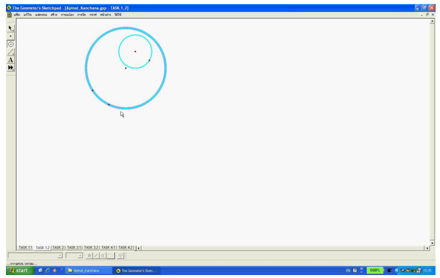
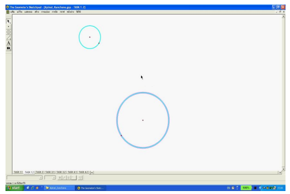
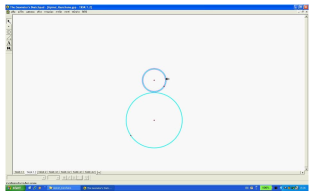
Table 9.9 Charles and Chloe's reactions to the Arrow tool

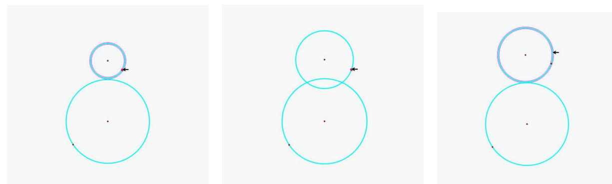
Line	Dialogues	Actions	Reasoning/Comments	Screen
	Researcher: What is the arrow tool for?			
	Charles: Let's see if it can be used to move that. (a straight line)		Charles abductively predicts the use of the arrow tool to move an object.	
		Chloe uses the arrow tool to select a straight line (figure a), then drag it up (figure b) and then down (figure c).		 (a) → (b) → (c)
	Chloe: Ahhhh! (pleasingly).			
	(7-sec pause)			
	Researcher: What is the arrow tool for?			
	Chloe: To move.			
		Charles then uses the arrow tool to adjust the size of a circle from the control point (figures d-f).		 (d) → (e) → (f)
	Charles: To move and to make it bigger or smaller.		Charles realises that arrow tool can manipulate	

Line	Dialogues	Actions	Reasoning/Comments	Screen
20	Charles: This one can adjust the degree. Chloe: The direction.	Charles then immediately moves to a straight line then tries to adjust its position (figures g-i).	figures in many different ways. Chloe may mean the line’s orientation.	 (g) → (h) → (i)

From this excerpt, Charles and Chloe consider the arrow tool to be a tool either to move the object around the screen (Line11) or to adjust the size or the orientation of the figure from a control point, i.e. a point on the circle's circumference or a point on a ray (Lines 11 & 22). These reactions suggest that students view the dynamic feature in the GSP as a tool to modify the constructed figure in various ways without a clear geometrical purpose. This notion becomes clearer when Charles and Chloe are encouraged by the researcher to draw any picture they like. They agree to draw a snowman which can be classified into the third group of 'Creative Objects' in Sub-section 9.1.5. Charles adopts the strategy that they can use the dynamic flexibility to 'draft' the picture first, then an arrow tool would help them adjust the draft to make the snowman look beautiful. The following excerpts show Charles and Chloe's strategy to draw a snowman using the dynamic feature in the GSP.

Table 9.10 Charles and Chloe's drawing of a snowman

Line	Dialogues	Actions	Reasoning/Comments	Screen
		Charles and Chloe agree to draw a snowman. Charles first constructs two circles at the top area of the sketch (figure a).		 (a)
	Chloe: Ow? (wondered) Charles: It can be moved.	Charles tries to adjust the size of the second circle.	Chloe wonders why Charles constructs two circles on top of each other while Charles is aware that the circles can be adjusted.	
	Chloe: How big should it be? Charles: Bigger is better. Make him big. Too small would leave too much space.			
		Charles then moves the two circles down one by one (figure b) forming a snowman shape (figure c)		  (b) → (c)
	Chloe: Yeah (pleased). Put them together.			

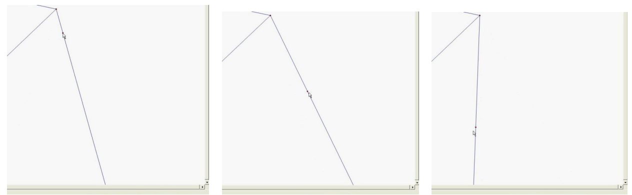
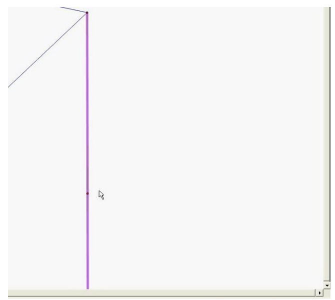
Line	Dialogues	Actions	Reasoning/Comments	Screen
20	Ah! The head is too small! Delete, delete, delete! Charles: Can we enlarge it? Yeah, we can enlarge.	Charles uses the arrow tool to enlarge the snowman's head (figures d-e) then move the head up so the two circles touch (figure f)	Chloe suggests Charles to delete the small circle supposedly to construct a bigger one.	 (d) → (e) → (f)

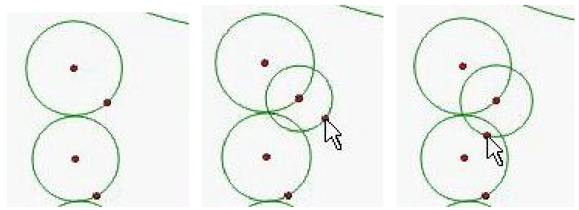
Charles and Chloe's strategy to construct a snowman illustrates that they interpret the feature of drag-mode in GSP as a tool to facilitate the drawing procedure. With the dynamic feature of the drag-mode, they can draw a picture leisurely then adjust it later to the desired shape. The fact that Charles draws two circles on top of each other at the top of the screen in order to determine the size of the two circles first (Lines 8-9, figure a) shows that he is aware of the flexibility in moving the figure around, while Chloe does not think of this in the first place. Charles also believes that he can readjust the size of the circle to make the snowman's head bigger by using an arrow tool (Lines 20-23, figure b), and move it in order to form the shape of a snowman (Lines 24-25, figure c). This interpretation of drag-mode use shows that students view the dynamic feature in GSP as part of the drawing tool to lend them to fix the figure with no geometrical concern at this stage. This preliminary interpretation of the function of an arrow tool is common across all the pairs of students interviewed.

9.2.2 Unique Portrayal of the Geometric Object

Apart from portraying a geometrical object by visible shapes as in the paper-and-pencil environment, the GSP also automatically adds an extra component to the constructed object, so that the user can use drag-mode to move the figure. Examples of these extra components are points on or at the end(s) of the straightedge objects, and a point on the circle's circumference. These extra components in the GSP are included to facilitate the user's manipulation of the constructed figure. They are not related to the concept of Euclidean geometry in any way. For this reason, their obvious presence in the GSP environment can confuse the students especially when they do not realise the raison d'etre of such components. The following excerpt shows students' responses reflecting this confusion.

Table 9.11 Brian and Bruce's reactions to the alien objects

Line	Dialogues	Actions	Reasoning/Comments	Screen
		Bruce tries the Ray command and moves the point on the ray around randomly (figures a-c)		 (a) → (b) → (c)
	Bruce: What is this?			
	Researcher: What did you say?			
	Bruce: What is this thing?		Bruce does not realise that it is a ray.	
	Researcher: What are you wondering?			
	Brian: We wonder why when we click, it becomes a line straight on			
	Bruce: And where does it end?		Bruce wonders about the boundary of the ray he constructed.	 (d)
	(3-sec pause)			
	Bruce: Oh, it lets you put the point here!	Bruce clicks to release the arrow. The point stays at the place (figure d).		
	Brian: Put points between the lines.		Brian interprets the tool	

Line	Dialogues	Actions	Reasoning/Comments	Screen
			as the point construction on the line	
	Bruce: Why point on the line?		Bruce wonders about the function of the point on the line.	
		Brian then takes turns to construct circles (figure e). When he constructs a circle using a point on the circumference as the centre (figure f), Brian moves the point on the circumference of the new circle around (figure g).		 (e) → (f) → (g)
	Bruce: Why is there a point here? Researcher: A point where? Brian: Here (on the circumference)		Bruce does not realise the function of the point on the circumference.	
		Brian uses his finger to		

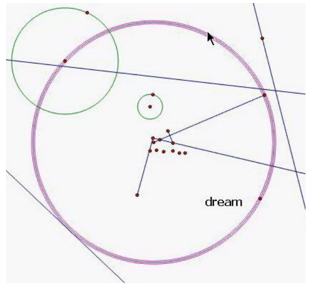
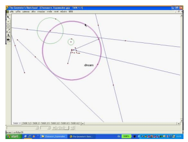
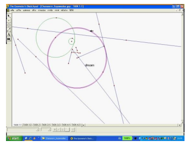
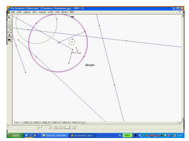
Line	Dialogues	Actions	Reasoning/Comments	Screen
		point at the point on a circle's circumference.		
	Bruce: You can also move that point on the circle.			
40	Brian: What is it? (laugh) I don't know!			

In this task, Bruce always asks questions when he encounters something that does not make sense to him. One of Bruce's most significant enquiries is the presence of points on the ray and the circle's circumference (Lines 20 & 32). Though he realises that these points can be moved with an arrow tool to adjust the construction as he desires (Lines 14 & 38), Bruce still wonders why the points stay when he clicks the mouse to complete the construction. He later realises that he can use it to drag the figure with the arrow tool. However, his uncomfortable preliminary reaction to these unfamiliar objects implies that these additional points should not be present in the finished construction. This may stem from Bruce's experience with the paper-and-pencil construction where these points would be redundant to the construction. This excerpt, therefore, shows how GSP's unique way of portraying Euclidean geometry to support software's dynamic feature can mislead the students' interpretation of their presence.

9.2.3 Discovery of the Parent-and-Child Relationship with the Drag-Mode

Besides the notion that the dynamic feature in the GSP can be used to move or adjust the size of the figure when the arrow tool is tried, two pairs of students also discover the parent-and-child relationship during the process, especially when they use the arrow tool to move a part of a complex construction. Such action results in an observation that some of the figure's components actually move in relation to each other. The identification of such relationship shows the students' awareness of the rule governed in the GSP construction, leading to the interpretation of the parent-and-child relationship in the DGS environment. The following two excerpts show students' discovery of, and their reactions to, the parent-and-child relationship in dynamic circumstances.

Table 9.12 Carla and Carol's discovery of the parent-and-child relationship

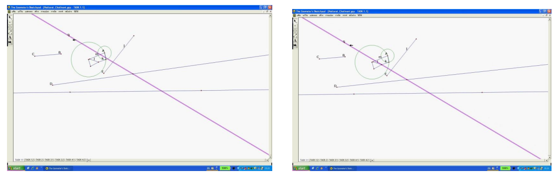
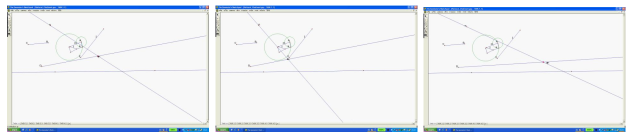
Line	Dialogues	Actions	Reasoning/Comments	Screen
		Carla first uses an arrow tool to select a circle.		
	Carla: Ahh, it covers. Researcher: What do you mean by cover? Carla: It's like, click . . . mmm, like when we draw a circle, when we click it, we can get the whole circle like this.	Carla uses an arrow to show the highlighted circumference.	Carla may notice that the circle is selected and something can be done with it.	 (a)
	(3-sec pause) Carla: I don't know whether it can be moved.		Carla abductively conjectures that the circle can be moved.	
	Carla: Yes, it can be moved. And the parts that connect to the circle also change. Researcher: How? Chloe: When you move this (the circle), the others also move.	Carla uses an arrow tool to drag the circle around randomly (figures b-d).		 (b) →  (c) →  (d)

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Carla: Lines or anything that is related, that is in the part of the circle will also move.			

From this excerpt, Carla first observes that the arrow tool can be used to select a circle implying that something can be done with the circle. She then adopts an abductive reasoning, possibly from her prior experience with the dynamic feature in GSP, to predict that the figure should also be movable by the arrow tool (Lines 10-11). This strategy of abductive reasoning may be categorised as under-coded abductive reasoning since Carla simply guesses the arrow tool's function (possibly from her experience with the software) without much confidence or logical explanation. When Carla finds out that the circle can actually be moved by the arrow tool she also notices that the connecting parts are also moving along with the circle. She identifies the parts that move along with the circle as 'related' to the circle in order to distinguish them from those parts that do not move along with the circle. Carla's response indicates that she appreciates the 'relationship' between the components of figures that influence the way the whole construction is moved under drag-mode.

Alex and Alan also observe this relationship when they try the arrow tool to drag the construction around. The following excerpt illustrates their reaction and interpretation of this relationship.

Table 9.13 Alex and Alan's discovery of the parent-and-child relationship

Line	Dialogues	Actions	Reasoning/Comments	Screen
		Alex uses an arrow tool to drag a straight line downward (figure a) then upward (figure b).		 (a) → (b)
	Alex: Ahh, to move the figure.			
	Alan: Mm? Between the point!		Alan notices that a certain part which has not been dragged also moves.	
	Researcher: What point?			
	Alan: Where the point, the connecting point. It affects each other.			
	Researcher: How?			
		Alan drags the intersection point of a straight line and a ray around randomly (figures c-e).		 (c) → (d) → (e)
	Alan: When we move one part, it will make the other parts move as well.			

From this excerpt, Alan first observes that the arrow tool can be used to move the figure around (Line 5). Nevertheless, Alex also notices that a certain part which is not dragged also moves along with the dragged part. He believes the intersection point which he calls a 'connecting point' (Lines 8-9) to be the factor that makes the parts move together. This reaction is similar to students' interpretation of the intersection point in the DGS environment as a point that 'glues' the construction together, as found in Jones' research (1996), though the relative movement is a result of the order of construction or parent-and-child relationship, not because of the intersection point.

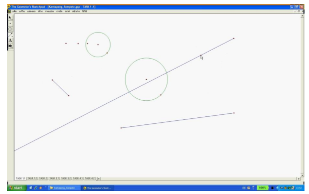
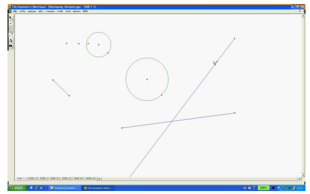
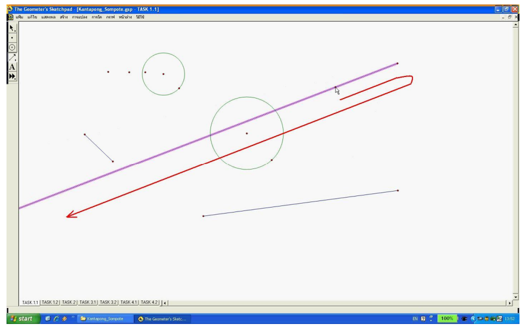
When Alan tries the arrow tool to drag the intersection point randomly around the screen, he explains that this is the result of an influence or 'affect' (Line 9) among different components. He finally concludes that when one part is moved, the programme will manage to move the other part as well (Line 11), giving the dynamic relationship between the figure's components under dragging.

These two excerpts clearly show the students' appreciation of components in the GSP environment and their relationship with each other. Such a relationship becomes visible when the drag-mode is used to move certain parts of the figure, provided the figures has a relationship with each other. The students' perception of the parent-and-child relationship in the GSP are expressed through keywords such as 'relate', 'connect' and 'affect'. Nevertheless, none of the students in the main study provide further explanation as to why the construction in the GSP behaves in such a way.

9.2.4 Dynamic Description of a Geometric Shape in the GSP

A curious view of how the GSP portrays Euclidean geometry is found in Colin and Conrad's interpretation of a ray. Instead of seeing it as a static entity, Colin perceives a ray to be a trace of a point that travels to one end and then back infinitely. This dynamic description of a static shape may indicate that the dynamic environment that students encounter in the DGS software influences their perception of how the geometric shapes are generated by the software. The following excerpt shows Colin's description of a ray in the GSP.

Table 9.14 Colin and Conrad's description of a ray in the GSP

Line	Dialogues	Actions	Reasoning/Comments	Screen
		Colin tries the ray command (figure a) then moves the ray around randomly (figure b)		 (a)  (b)
	Colin: Ha!? (amazed) Confused!			
	Researcher: What confuses you?			
	Colin: Why it springs out through the mouse position here.		Colin wonders why a ray is longer than the mouse arrow.	
	Conrad: Oh, it reverses back.			
	Researcher: How does it reverse?			
		Conrad uses his figure to point the ray from an arrow position to one end then backwards (shown with a reversed arrow in figure c)		 (c)

This response shows that neither Colin nor Conrad is familiar with the concept and definition of a geometrical ray. This may be the reason that they wonder why the ray is extended beyond the mouse position (Lines 7-8). From this unfamiliarity, Conrad gives an astonishing observation that the constructed object may be a trace of a point that travels from the arrow point to one end, and then 'reverses' (Line 9) the same path through the original point infinitely. This dynamic description of a static figure can imply the influence of the dynamic environment to make the student abductively conjecture that the a static figure is actually a trace of a travelling point, similar to a case where a circle is a trace of a pencil stick to a set of compasses which Conrad described previously when they tried the Compass Tool command. Alternatively, it can be an influence from the construction in the paper-and-pencil environment where lines are drawn gradually in a certain direction, which is a dynamic process even though the end product would become static. This possible influence highlights the dynamic activity involved in a static figure construction in the traditional paper-and-pencil environment which the student can adopt to interpret the object in the GSP.

9.3 DISCUSSION OF STUDENTS' INTERPRETATION OF THE GSP ENVIRONMENT

Using the analysis model in Figure 9.1 to analyse these students' reactions to the GSP features with and without the dynamic feature in Task 1, the tensions between each relationship can be outlined. One of the important factors that appear to have a strong influence on students' interpretations of the GSP environment is the way GSP portrays Euclidean geometry. It can be seen from the excerpts in this chapter that GSP's portrayal of Euclidean geometry is based on the paper-and-pencil environment, with the enhancing feature to portray the geometric objects more theoretically accurately, such as the infinite plane or a ray and a straight line, as well as the additional features such as the control objects or parent-and-child relationship to allow the dynamic flexibility of the construction. These features distinguish the DGS environment from the paper-and-pencil environment leading to a new and unique way of Euclidean geometry representation.

However, all the participant students in this research have experience with Euclidean geometry through the paper-and-pencil environment. They appear to use this experience as a basis to make sense of the commands and behaviour in the GSP. Some students, therefore, get confused with these enhancing and additional features in the GSP, especially when they see the objects appear and behave differently from their more familiar paper-and-pencil environment, as discussed in Sub-section 9.1.1 and Sub-section 9.2.2, while some students manage to make sense of these. This tension relates to the relationships 3-5 and 4-5 in the analysis model where students need to use their prior knowledge of geometry to familiarise themselves with the GSP feature and its dynamic portrayal of Euclidean geometry.

Apart from their prior knowledge of Euclidean geometry, students also have a prior perception of the usefulness of this subject. This perception appears to influence the way students perceive the benefits of the GSP as a tool and generate a process of command justification where they need to understand each command's purpose in order to understand its use. This again may cause tension when the student cannot realise some of the commands' practical usefulness, as presented in Aaron and Anna's case in Sub-section 9.1.2. The justification of the GSP as a useful tool becomes more apparent when students are invited to use the software to draw anything they like. The students' diverse approach to application of the GSP categorised in Sub-section 9.1.5 also shows how they perceive the usefulness of the Euclidean geometry subject. This in turn may influence the way students interpret the usefulness of the dynamic feature, where some of them consider the dynamic flexibility in the GSP to be a facilitating tool to help them adjust the picture in the way they wish. Such perception is illustrated in the cases of Colin-Conrad and Charles-Chloe in Sub-section 9.2.1. This part of the reaction also involves the relationships 3-5 and 4-5 in the analysis model.

For the relationship 3-4, where the students are supposed to interpret the way the GSP functions in order to portray Euclidean geometry in dynamic mode, involving the parent-and-child relationship, some students appear to be able to make sense of such dependent relationship and recognise that one object in the GSP environment may depend on the existence of another, or may move in relationship to each other, as shown in Sub-section 9.1.3 and Sub-section 9.2.3. However, the other part of the software rule that depends on the designer's choice, such as the object-before-command algorithm, can cause confusion to the students when they cannot figure out the designer's intention of how to execute a particular command, as depicted by Bob and Bridget's case in Sub-section 9.1.4.

Command justification becomes a key reasoning activity in Task 1 when students need to learn to use the command by trying to understand its function. Apart from knowing what each command can do, some students also feel the need to understand the raison d'etre of each command, i.e. what it is actually used for in a practical sense. This reaction may stem from students' prior perception of the practical use of Euclidean geometry knowledge, forcing them to find the reason to explain why the command is included in the GSP. In order to make sense of each command, students appear to adopt a range of reasoning strategies. Inductive reasoning is used to generalise the common property of the command's output to learn what the command can do. Abductive reasoning is also used to predict a particular command's' function based on the similarity to other commands students used previously. Deductive reasoning is also apparent when students need to draw on the Euclidean geometry rules and properties learned in order to interpret the GSP environment, which turns out to be one of the most important factors, especially when the students' knowledge is not accurate.

The analyses and discussions of students' reactions to Task 1 in this chapter illustrate the tensions between three entities in the model of study: the Learner, the DGS and Euclidean geometry, when students experience the GSP environment for the first time. These tensions in the entities' inter-relationships provide a basis for the answer to the research sub-question 1 which will be discussed again, together with other research questions in Chapter 11.

10 LEARNERS' REASONING IN THE DGS

This chapter presents the data analysis of learners' reasoning strategies found during the task-based interview. The researcher examines learners' responses during Tasks 2-4 of all nine pairs of participant students from the main study, and then looks for the different types of reasoning students adopt during the tasks. In these tasks, the reasoning activities are identified by the students' response to the question of why certain geometry property is observed or needs verification. They are categorised into inductive, deductive and abductive reasoning as defined from the Literature Review chapter. This part of the students' response forms the inter-relationship 1-3-5, 1-4-5 and 2-3-5 from the analysis model used as a basis for the research sub-question 2: "What kind of reasoning strategies do learners adopt in geometric construction and exploration tasks in the DGS environment?"

The inter-relationships 1-3-5, 1-4-5 and 2-3-5 are presented with thick arrows bordering the highlighted area in Figure 10.1.

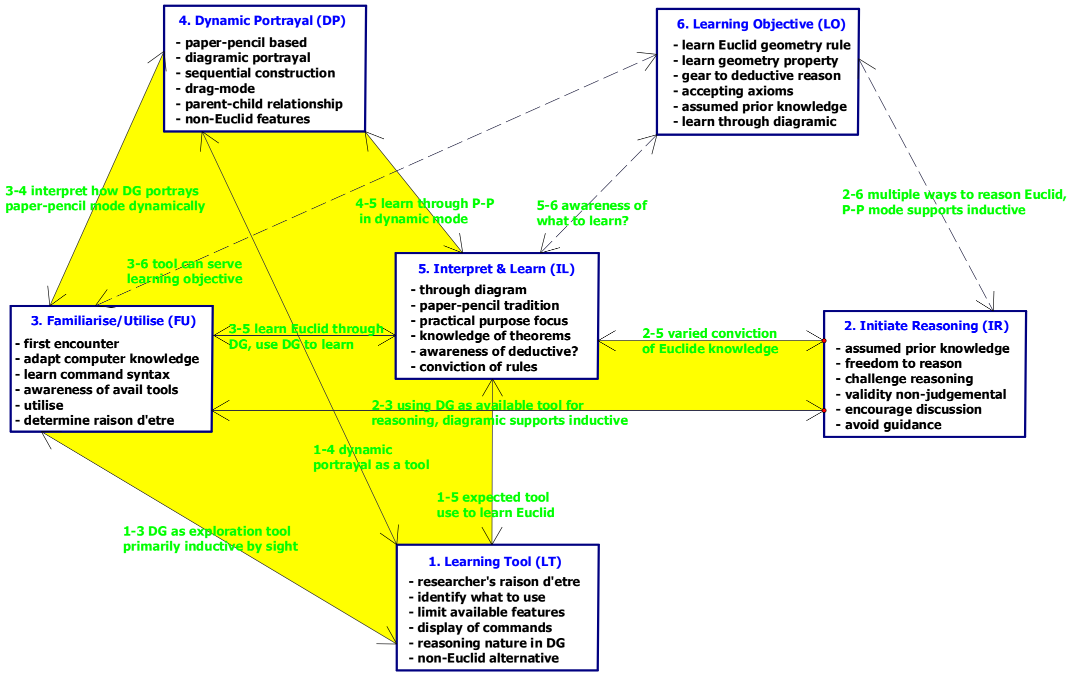


Figure 10.1 Inter-relationship 1-3-5, 1-4-5 and 2-3-5

The learners' reasoning strategies found in this part of research are categorised into two groups. The first group consists of reasoning, which is directly related to some features of the DGS environment and is depicted as inter-relationships 1-3-5, 1-4-5 and 2-3-5 in the model of study. The second group consists of reasoning, which is not directly related to any of the features of the DGS environment. This group of reasoning strategies therefore involves only the relationship 2-5, between the reasoning-challenge approach of the design tasks and the students' reasoning, with no direct relation to the relationship 1-Learning Tool or 3-Familiarise/Utilise. Though this second group of reasoning strategies do not reflect the significant role of the DGS as a tool, it is still relevant for inclusion in this chapter since it helps the researcher to compare and contrast the reasoning styles, with and without the aid of DGS features. These two groups of reasoning strategies are presented in the following two sections. The chapter concludes with a discussion of all reasoning strategies found in this part of the research.

10.1 REASONING RELATED TO DGS FEATURES

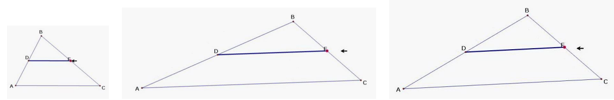
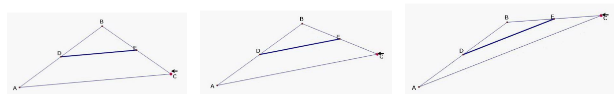
This section discusses students' reasoning strategies found in the interview which are directly related to a particular GSP feature, as well as the categorisation of reasoning strategies based on the axial codes identified in Sub-section 5.7. It reflects the inter-relationships 1-3-5, 1-4-5 and 2-3-5 in the analysis model where learners are assumed to use DGS as a learning tool to help them reason.

10.1.1 Reasoning with a Reference to the Dynamic Feature

The prevalent role of the dynamic feature of drag-mode in the GSP found in this research allows students to observe the 'variant' and 'invariant' properties of the figure under dragging. All the participant students in the main study can observe the intended properties in Task 3; the fact that the perpendicular bisector of a chord always passes the circle's centre in Task 3.1; and the parallel property of the triangle midpoint theorem in Task 3.2 after dragging, though some of them may need the researcher's guidance to concentrate on a particular element or property. Since this way of observation still relies on perception by sight, it should be categorised as inductive reasoning. The main strategy is therefore to distinguish the variant and invariant properties of the figure in the dynamic circumstance, by the process of generalisation and discrimination of the object's attribute as suggested by Klauer (1996) in Sub-section 2.4.2. An example of this inductive reasoning strategy is given in Alex and Alan's reactions to Task 3.2, presented in Excerpt 8.1 in Chapter 8, where the issue of reliability of observation by sight, even in the dynamic situation, is discussed by the pair of students.

However, there is also a circumstance where the student explicitly incorporates the drag-mode in the GSP environment in their justification process. The following excerpt is taken from Task 3.2, conducted by Aaron and Anna when they are asked to explain why the segment connecting midpoints DE is always parallel to the side AC.

Table 10.1 Aaron and Anna's induction from the dynamic feature

Line	Dialogues	Actions	Reasoning/Comments	Screen
		Anna drags point E to the right (figure b) and then back (figure c).		 (a) → (b) → (c)
	Aaron: DE is parallel to AC, isn't it?		Aaron observes the midpoint theorem	
	(5 sec pause)			
	Aaron: They go together.			
	Researcher: What if we drag other points?			
		Aaron drags point C up (figure e) and then to the left (figure f).		 (d) → (e) → (f)
	Aaron: They are still parallel.			
	Researcher: Why?			
	Anna: Because when you move, they are still parallel.		Anna inductively reasons by asserting that the parallel property remains when things are moved.	

From this excerpt, Anna justifies the parallel property by asserting that DE and AC remain parallel to each other when the figure is moved by drag-mode (Lines 13-14). Though she inductively verifies this by empirical observation that the segments are always parallel, she explicitly claims that the statement is valid under a particular circumstance, i.e. when the figure is moved.

This means of justification is more sophisticated than simply verifying that 'they remain parallel' or 'always parallel', as exemplified in Alex and Alan case in Excerpt 8.1, since the way in which the figure is moved is explicitly used in part of the reasoning statement (Line 13). Anna's response in this excerpt suggests that she considers the motion in the GSP environment as a criterion to verify a constant geometric statement, together with her observation by sight (seeing that they are parallel). The retention of the figure's particular property under movement is therefore used as a 'warrant' to verify the claim, similar to Alex and Alan's case, presented using Toulmin's argumentation model in Error! Reference source not found..

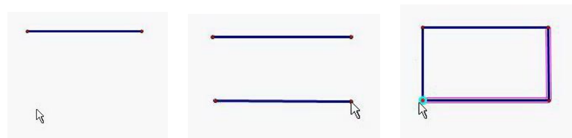
This reaction is also similar to the student's response in Leung's study (2009) discussed in Section 2.9, where the student explicitly uses a prime sign and an equation form to indicate that a certain property is true when the element (with the prime sign) is dragged. The dynamic feature of the drag-mode in the DGS can therefore play a role in ascertaining students' empirical observation, and used by students as a tool to strengthen justification of the statement. This reflects the inter-relationship 1-4-5 in the analysis model, where the dynamic feature in the GSP is explicitly used by the students to confirm the Euclidean geometry rule.

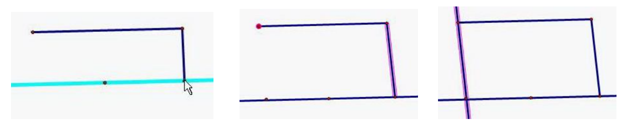
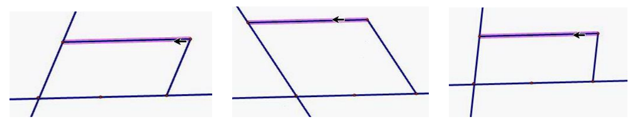
10.1.2 Reasoning from the Process of Construction with the DGS

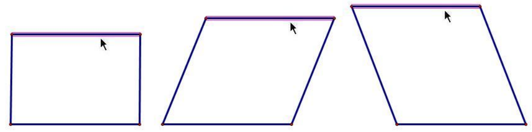
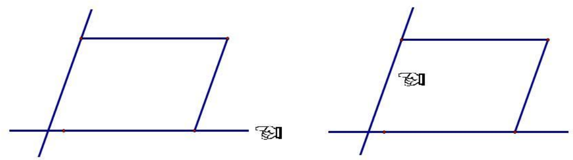
A common reasoning found in many pairs of students is the reasoning by deduction from the process of construction in the DGS environment. In this instance, students value the process of construction with the DGS command as a reliable factor in making the pertinent figure possess the assumed properties. During Task 2, where students are supposed to construct a robust parallelogram, Barbara and Beth adopt two approaches in sequence. Barbara first suggests trying the ordinary way, i.e. lining up four segments by sight to form a parallelogram which she ends up constructing in a rectangle-like shape. Then the pair move on to use the Parallel Line command to construct a parallelogram, though the fact that the Parallel Line command produces an infinite line makes Beth question whether it is the correct approach.

The students perform the drag test with both figures, but because they drag the figures' sides instead of vertices, both figures remain a parallelogram when the sides are moved. Barbara suggests that the second approach should be more precise and acceptable, since they use the Parallel Line command, while the first approach is non-mathematical and should be deleted. This behaviour shows how students value the mathematical process involved in geometric figure construction, even though the output of both approaches appear to resemble a parallelogram, especially when they both pass the drag-test. The students finally justify their construction by deductively referring to the use of the parallel line command. The following excerpt shows Barbara and Beth's approach to the parallelogram construction.

Table 10.2 Barbara and Beth's deduction of the process of construction

Line	Dialogues	Actions	Reasoning/Comments	Screen
		The researcher asks the students to construct a parallelogram then leaves.		
	Barbara: Let's try the ordinary way.	Barbara uses the Segment tool to construct a parallelogram by sight without the Parallel Line command (figures a-c)		 (a) → (b) → (c)
	Beth: What if they are not parallel? Barbara: Just try first. Beth: No, it's not!		Beth argues that this method cannot confirm the parallel property.	
		Beth then tries to construct with the Parallel Line command (figures d-f).		 (d) → (e) → (f)
	Beth: Mmm? Is it too long? Barbara: It's okay, let's go on. (15-sec pause)		Beth refers to the infinite parallel line.	

Line	Dialogues	Actions	Reasoning/Comments	Screen
		Beth draws the right-side segment (figure g) then constructs another parallel line with that side (figures h-i).		 (g) → (h) → (i)
	Barbara: Parallelogram does not need to be perpendicular right?	Beth performs drag-test by dragging the top segment up (figure k) and down (figure l)	Barbara suggests that the right side does not need to be perpendicular to the parallel lines.	 (j) → (k) → (l)
	Beth: Okay, parallel! Barbara: Yes! Beth: Okay. (4-sec pause)			
	Beth: But there are redundant lines Barbara: Let it be!	Beth drag-tests the first	Beth means the extended sides of the parallelogram.	

Line	Dialogues	Actions	Reasoning/Comments	Screen
		construction by dragging the top segment as well (figures m-o).		
	Beth: Oh? This one is also a parallelogram!			(m) → (n) → (o)
	Barbara: But the new one is more parallel. That one we just messed around. Just delete it!		Barbara rejects the first solution though it appears robust.	
		Barbara deletes the first parallelogram.		
		(The researcher returns)		
	Researcher: Okay, why do you think this is a parallelogram?			
	Beth: We used the Parallel Line command, this side (the bottom side-figure p) and this side (the left side-figure q).	Beth physically points at the two constructed parallel lines.	Beth verifies the parallel property from the construction command.	 (p) → (q)

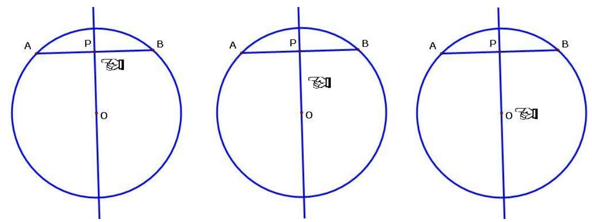
From this excerpt, Barbara and Beth adopt two different approaches to construct the parallelograms which, under their means of drag-test (dragging the sides rather than vertices), each construction appears to remain a parallelogram (Lines 29 and 39). Nevertheless, when they have to judge which one they would use as the task solution, Barbara reasons that the parallelogram constructed with the Parallel Line command should be the correct solution. She values the mathematical process in construction over the non-mathematical process, even though the outcomes are indifferent. This can be deemed to be a deductive reasoning process, based on their trust in the DGS command; that it should give more precise construction than free-hand drawing.

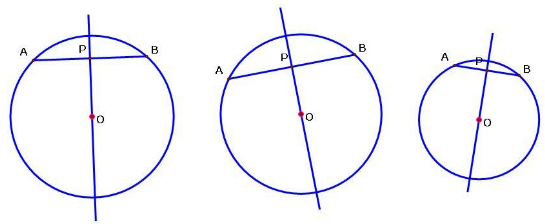
When asked why the construction is a parallelogram, Beth makes an explicit claim that the figure is constructed using the Parallel Line command (Line 48), to justify the figure as a parallelogram. This justification may be considered a deductive reasoning, where the fact that the Parallel Line command in the GSP produces accurate parallel lines is used to confirm the parallel property, reinforcing their preference of the construction by geometric property over the construction drawn simply by hand.

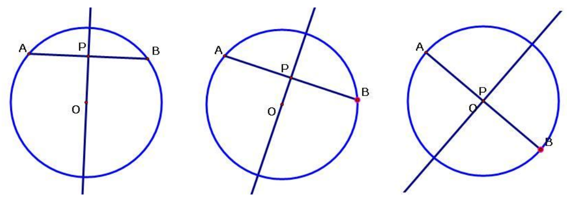
Note that the reason that these two constructions behave indifferently under the drag-test is because the students choose to drag the figure's side rather than the vertex. When a segment is dragged, it is simply translated around the screen, pulling the connecting side in the same direction, resulting in parallel movement. Had the student dragged the vertex of the construction, the first construction drawn by free-hand would have behaved differently. The observed invariant parallel property of the first construction is therefore as a result of coincidence. The figure is not truly robust to all kinds of dragging.

Another instance of reasoning from the process of construction is the deduction of the construction itself during the justification process. Though this way of reasoning may not give any further information since the inference is based on the instruction of the task itself, it is still a valid reasoning and can be viewed as a method students use to reflect the relationship between the process of construction and the observed behaviour. The following extract is taken from Task 3.1 by Aaron and Anna when they start to explore the constructed perpendicular bisector of the chord.

Table 10.3 Aaron and Anna's deduction to the process of the construction

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Researcher: How about the line we construct? Aaron: It's perpendicular. Researcher: Yes, but that's because we construct it as perpendicular. Anna: It will halve. Aaron: Yeah, splits the circle into two halves. Because we draw this line through the midpoint so it will be half. And P and O are on the same plane. Anna: When you halve P, it's the same as halving O. Researcher: Did we construct the line to	Aaron physically points along the perpendicular line from P (figure a) down (figure b) to O (figure c).	Anna spots the symmetric property. Aaron uses the term 'on the same plane' though he possibly means 'aligned'. This conclusion may stem from the fact that the students see that the perpendicular line always passes through point O.	 (a) → (b) → (c)

Dialogues	Actions	Reasoning/Comments	Screen
pass point O? Aaron: No. Anna: But P is the midpoint. So it passes point O. Researcher: Does it always pass point O? Aaron: Should be because it is the midpoint of the chord. Aaron: When you move point O, the line follows but P remains the midpoint … because it is a midpoint! Researcher: So why when we construct a perpendicular line through the midpoint of	Aaron drags point O around briefly (figures d-f).	Aaron gives the reason that the line passes point O because P is the midpoint of the chord. Aaron drags-tests the figure and observes that P remains the midpoint. He realises by himself that the reason is that it is constructed as a midpoint.	 (d) → (e) → (f)

Line	Dialogues	Actions	Reasoning/Comments	Screen
	the chord, it would pass through point O? Anna: Maybe because the chord is similar to the diameter of the circle. When we move point B, they will move like this. So they are similar.	Anna moves point B down (figure h) until point P is on point O (figure i)	Anna abductively relates the property of the circle's chord and the diameter to show that the midpoint P is similar to the midpoint of the diameter.	 (g) → (h) → (i)
	Researcher: How? Anna: This is the chord. When we move it here (figure i), it can be the diameter. So they are similar.		Anna uses the drag-mode to demonstrate the similarity.	

Since Aaron and Anna do not manage to observe that the perpendicular bisector of the chord always passes through point O, the researcher has to guide them towards the constructed perpendicular line (Line 1). Aaron first asserts that the line is perpendicular (to the chord AB) at Line 3, which is an obvious property since the line is constructed using the Perpendicular Line command. Anna first relates the perpendicular line to point O by saying that the line halves the circle (Line 6), and Aaron elaborates Anna's observation by claiming that P and O are on the same plane (Lines 9-10). However, the way he points at the construction in figures a-c suggests that Aaron actually means P and O are aligned by the perpendicular line. This reaction indicates that the students are now aware that the perpendicular line always passes through point O.

However, when the researcher challenges this pair of students to justify their observation, both Anna and Aaron explicitly reason that it is because P is the midpoint of the chord (Lines 20, 23-24), which is actually the instruction of the task. This claim illustrates a deductive reasoning based on the process of the construction used in the task. Note that Aaron also uses the dynamic feature of the drag-mode to move point O around, possibly to try to pull point O out from the perpendicular line. He observes that the line follows point O and point P remains the midpoint (Line 27-28) and realises later that it is because it is constructed as a midpoint at Line 29. In fact, the perpendicular line moves to keep the perpendicular property following point P, which is dependent on the chord AB moving along with the circle when point O is dragged. Aaron, therefore, has a reverse interpretation of the parent-and-child relationship in this circumstance, believing that the perpendicular line follows point O. This reaction is similar to the finding in Talmon and Yerushalmy's study (2004) discussed in Sub-section 2.8.2, where the student expects the reverse order of dependence from how the construction is organised in the DGS environment. Aaron's statement that P remains the midpoint because it is 'a midpoint' at Lines 28-29 suggests his appreciation of the unchanged property set by the construction command in the GSP, regardless of how the figure is manipulated.

Though the students' deduction in the construction process does not provide any additional information for justification, it still helps the students to identify the important property that makes the figure behave in such a way, especially when such property would be persistently retained in the dynamic environment. This becomes evident when Anna tries to adopt abductive reasoning to compare the midpoint property of point P and the midpoint property of point O on the circle's diameter. She uses the drag-mode to drag point B until the chord appears to be the circle's diameter (Line 47, figure i). Note that the word 'similar' used by Anna at Lines 37, 40 and 48 suggests that she believes the circle's chord and circle's diameter to be two different entities. Though her expression that the chord can be the circle's diameter at Line 47 may imply that she sees the circle's diameter as one of the chords, her emphasis on the 'similar' property of the midpoint P and O suggests that her main reasoning strategy is to find the connection between P as the midpoint of the chord and O as the midpoint of the circle's diameter, despite her failure to explain the exact relationship. Her reasoning therefore remains at an abductive level where the common property of the midpoint P and O is used to verify the observation. Nevertheless, neither Aaron nor Anna uses the perpendicular property of the construct line in this latter stage of reasoning. They only concentrate on the midpoint property and are not able to verify the observation fully with deductive reasoning based on these two key properties.

Aaron and Anna's deduction in the construction process of the midpoint P in this task leads them to the next stage of abductive reasoning with the common property. This kind of reasoning, though not very productive by itself, may provide a helpful stepping stone to further reasoning.

10.1.3 Multiple Cases of the Same Figure

Another obvious advantage of the dynamic feature of the drag-mode in the GSP is that it allows the students to adjust the shape into a form to help them reason. This flexibility helps students to relate the geometry property they know, to the observed property, in order to find the explanation. This leads them to adopt an abductive reasoning strategy by finding a rule that may be applicable to the case observed.

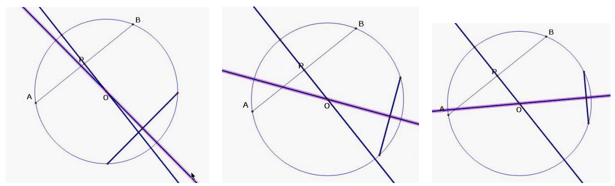
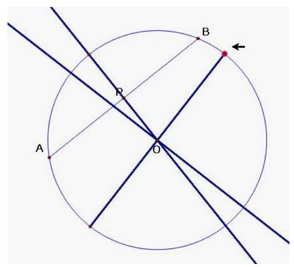
Examples of this instance are given in Alex and Alan's use of the collapsed figure in order to relate it to the parallel property in Excerpt 8.1, and Barbara and Beth's adjustment of the triangle into a right angle triangle in order to relate it to the constant ratio property of a right triangle's three sides they learned from the Pythagorean lesson in Excerpt 8.8. These examples illustrate how the dynamic flexibility in the DGS defines shapes differently from the static paper-and-pencil environment. A triangle can now be adjusted to look like collapsed segments or a particular kind of triangle such as a right angle triangle. (Arguably, the DGS may define a right angle triangle differently, i.e. to be constructed with the Perpendicular Line command.) However multiple possible appearances of the same object lend themselves to different interpretations of a shape from the paper-and-pencil mode. The distinction between the acute triangle, the obtuse triangle and the right angle triangle is now blurred since a generic triangle in the DGS environment can be visually transformed into any of these by simply dragging just one point. The dynamic flexibility, therefore, raises an issue about the new perception of shapes in the DGS environment. This directly affects how students define geometric shapes and the way in which they relate one kind of shape to another. It complicates the van Hiele's proposal of levels of geometric understanding, where the common property of a shape such as a generic triangle needs to be clearly distinguished from the properties of other particular shapes, such as a right angle triangle or an isosceles, apart from the distinction between the particular shapes. The fact that DGS visually adopts the inclusive geometric definition, e.g. an equilateral triangle is also one type of an isosceles, also forces the students to appreciate such an inclusive definition in order to make sense of the shape's behaviour in the DGS.

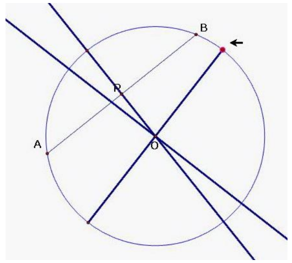
So even if the flexibility of the dynamic environment allows them to start to reason from the particular appearance of the shape, students still need to be able to connect such reasoning to a general case to ensure that the explanation is also applicable to the generic shape. Such a challenge can help to refine students' understanding of geometric shapes by distinguishing between properties belonging only to a particular shape and those which belong to the generic shape.

As a result of this, when dealing with a generic shape such as a triangle in Task 3.2, students usually utilise 'Guided dragging' in order to adjust the generic triangle into a particular triangle. Students' utilisation of drag-mode in this research has a distinct characteristic from the dragging schemes such as 'Bounded dragging', 'Dummy-Locus dragging' or 'Line dragging', as identified by Arzarello, Olivero et al. (2002). While Arzarello, Olivero et al.'s categorisation of these dragging schemes is concerned with how a single point in the figure is dragged, students in this research focus more on how the shape changes under dragging. The dragging characteristic of students in this research therefore relates more to the appearance of the shape, and not how a particular point is moved. This may be the result of the different level of geometric understanding as proposed by Van Hiele (Usiskin, 1982). When dealing with a particular shape, most students in this research perceive the shape as a whole (Level 1 of original van Hiele model), rather than the property of the figure's smaller components (Level 2 of the original van Hiele model), as in Arzarello et al.'s studies (1998; 2002). This can be seen from Rachel's identification of a 'reversed triangle' in Figure 7.13 in the pilot study or in the cases of Alex-Alan and Barbara-Beth in Excerpts 8.1 and 8.8 respectively as mentioned above.

Apart from analysing the geometric shapes' property from multiple cases with the drag-mode, some students enhance their inductive reasoning strategy by reconstructing the additional elements from the instruction, giving another case in the same diagram in order to confirm the discovered property. With this additional case, the students have an opportunity to view two different circumstances in the same figure at the same time. This can lend them to compare and contrast the two cases. Moreover, they also have an opportunity to compare and contrast the two examples in the static and dynamic modes by dragging one of the constructions. The additional case in the dynamic environment therefore allows students to consider the task in a subtler way other than inductively examining two cases. The following excerpt shows how Alice and Alma gain an insight when they construct an additional case of a perpendicular bisector of the chord in the same circle in Task 3.1.

Table 10.4 Alice and Alma's additional case construction

Line	Dialogues	Actions	Reasoning/Comments	Screen
		After observing that the perpendicular bisector of the chord AB always passes through point O, Alma tries to construct a new chord and repeat the construction (a) then drags the new perpendicular line to the right (b), (c)	Alma uses another example and drag-mode to inductively verify the property.	 (a) → (b) → (c)
	Alma: This line also passes through O. Researcher: Yes, but why? Please discuss to find the reason.	(The researcher leaves) Alma deliberately moves the new perpendicular line up so that the new chord becomes a diameter (figure d).		

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Alice: When it's perpendicular to the chord, that means the diameter . . . Let's say, the midpoint of a chord and a perpendicular line will pass the circle's centre because it is also a diameter, isn't it? Is this a reason?	Alice turns to ask Alma. (The researcher returns)	Alice tries to relate the chord to the circle's diameter.	(d)
	Researcher: Do you get the reason? Why did you move this chord here?	Both smile shyly		
	Alice: Because it is a midpoint of the chord when we move it, it can be a centre of the circle.		Alice relates the chord's midpoint to the circle's centre.	 (d)

From this excerpt, Alma constructs a new chord and the perpendicular bisector of that new chord in the same circle. She uses the dynamic feature of the GSP to drag the perpendicular line to the right before concluding that the perpendicular line also passes through point O (Line 10). This illustrates an inductive reasoning strategy with Naïve Empiricism, where two cases in the same circle are used to confirm the observed property, together with the invariance of such property under dragging.

After this inductive phase of reasoning, Alma moves the new chord up in order to make it look like a diameter of the circle (Lines 15-16) showing two simultaneous contrasting cases; one when the chord passes through the circle's centre and becomes a diameter (the new construction); and one when the chord does not pass through the circle (the original chord AB). Figure d in this excerpt therefore provides two snapshots of the chord's transformation, from a smaller one with a midpoint P, to the biggest chord possible when the chord passes through the centre O turning itself into a diameter. The visual presentation of these snapshots may help Alice to relate the circle's chord to the circle's diameter, where she explicitly claims that the chord can also be a diameter (Line 22). The midpoint of the chord can therefore become a midpoint of the diameter, which is the centre of the circle (Lines 28-29). This reasoning implies that the perpendicular bisector of the chord needs to pass the centre O so the chord can become the diameter of the circle.

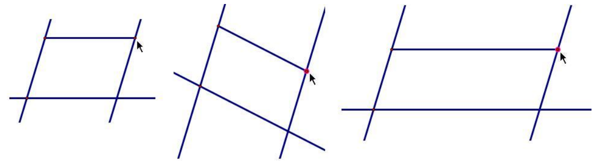
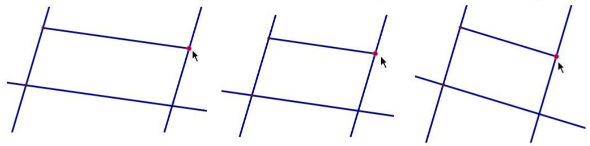
Since Alice never refers to the geometric definition or property of either the chord or the diameter, this reasoning remains at an abductive level where she assumes that the chord of the circle can become the circle's centre from her observation of these two cases in the same figure. She appears to appreciate the inclusive definitive nature of the DGS when she learns that the circle's chord can be transformed into a diameter in the DGS environment. This acquisition of the inclusive definition concept shows the influence of the Dynamic Portrayal relationship (Box 4) on the analysis model in Figure 10.1 and how it affects the way students Interpret & Learn (Box 5) the Euclidean geometry concept when DGS is used as a Learning Tool (Box 1). The connection between the chord's midpoint, and the circle's centre as the midpoint of the diameter, is not possible if students do not realise that the circle's diameter is actually a particular type of chord as portrayed by the GSP.

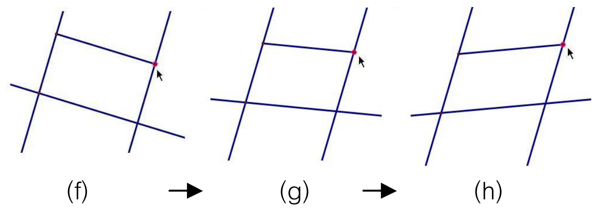
10.1.4 Reasoning from Partial Understanding of the Geometric Shapes

One of the interesting tensions when students deduce their prior knowledge of geometry in the DGS environment is the conflict between their partial understanding of geometric shapes and the way DGS portrays such shapes. This tension includes the clash between the DGS's inclusive definitive characteristic of the shape's portrayal, and their exclusive understanding of geometric shapes, or fixation to the traditional orientation of a particular shape as experienced in the paper-and-pencil environment, leading to confusion when they see the flexible shape. These issues are discussed and illustrated by the students' actual responses as follows.

With the fact that the DGS environment visually adopts the inclusive definition of the geometric shapes, conflicts may arise when students show an exclusive definition understanding of a geometric shape: for example a rectangle is not considered as a subset of the parallelogram. Examples can be applied from students' performances on Task 2 when they are asked to construct a robust parallelogram. Some students may have uncertainty when the constructed figure is dragged into some undesired shape, misleading them to think that the construction might be wrong. The following excerpt shows how Alex gets confused while he tries to deductively justify the constructed parallelogram.

Table 10.5 Alex and Alan's exclusive definition of the parallelogram

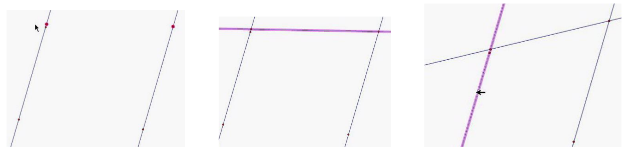
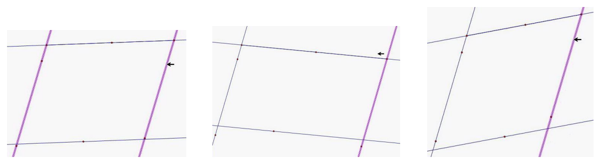
Dialogues	Actions	Reasoning/Comments	Screen
	Alex and Alan complete the construction with the Parallel Line command and try to drag-test the figure. Alex drags the top-right vertex down (figure b) and then up and to the right (figure c).		 (a) → (b) → (c)
Alan: Is this alright? (he asks the researcher) The shape changes but it remains a parallelogram. Researcher: How? Alan: It gets smaller but still it is . . .	Alan turns to ask the researcher.	Alan inductively justifies the construction by sight since it remains a parallelogram under drag.	
	Alex continues to drag the vertex until he notices that it appears like a rectangle (figures e-f).		 (d) → (e) → (f)

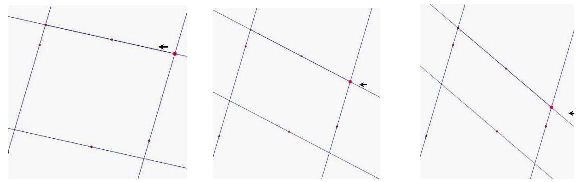
Line	Dialogues	Actions	Reasoning/Comments	Screen
	Alex: Look! A rectangle. Obviously! (laughs shyly) Is it a parallelogram? It's no longer a parallelogram! Alex: Now parallelogram.	Alex deliberately drags the figure until it looks like a parallelogram again (figures f-h) without further discussing the rectangle he finds.		

From this excerpt, Alan is convinced that the construction is a parallelogram since the parallel property is retained regardless of how the figure's appearance is changed, i.e. getting smaller (Line 13). This response shows that the drag-test to show the invariant property of the parallel sides helps Alan ascertain that the construction is correct despite his request for confirmation from the researcher (Line 9). Nevertheless, when Alex drag-tests and finds that the parallelogram can also become a rectangle, he suddenly expresses doubt as to whether the construction is still a parallelogram or not (Lines 18-19). This reaction depicts Alex's appreciation of the exclusive definition of a parallelogram, which he clearly distinguishes from a rectangle, considering them as belonging to the different categories. Alex's exclusive definition understanding of geometric shapes disturbs his justification process in the parallelogram construction task. The encounter with the rectangular shape provides a counter-example of how Alex may inductively negate the claim that the construction is always a parallelogram. This tension shows how the student's prior understanding of geometric shapes from past lessons may influence their reasoning process to justify the task, even though they adopt the correct strategy to construct the given figure. This confusion is also found in Jones' research (2000) where students are to classify quadrilaterals in the DGS environment, and the problems faced when their adherence to the exclusive definition understanding of quadrilaterals prevents them from realising the inclusive definition portrayal of the DGS.

Another example where the DGS portrayal of geometric shapes may contradict students' geometric understanding can be found in Carla and Carol's case when they try to justify the constructed parallelogram. In this case, Carla falsely considers a parallelogram with its top and bottom sides lying non-horizontally is not a parallelogram. This reaction shows Carla's adherence to the traditional presentation of a parallelogram in the paper-and-pencil environment, where one of the sides is usually drawn in a horizontal orientation. The following excerpt illustrates Carla's reaction to the constructed parallelogram.

Table 10.6 Carla and Carol's exclusive definition of the parallelogram

Dialogues	Actions	Reasoning/Comments	Screen
	After constructing a pair of parallel lines (figure a), Carla uses the Straight Line tool to construct the top side of a parallelogram (figure b) then performs drag-test with one of the parallel lines (figure c).		 (a) → (b) → (c)
Carla: No, that's not. It's difficult. I just got parallel lines. But when I draw this straight line and drag here it's no longer aligned. The angle changes . . . Can the angle change? Researcher: Is it a parallelogram? Carla: Maybe it can change.	Carol then helps Carla to construct the bottom side using the Parallel Line	'aligned' is translated from a tricky and ambiguous word in Thai, it should mean 'horizontal' or 'aligned to the horizontal line' in this case.	 (d) → (e) → (f)

Line	Dialogues	Actions	Reasoning/Comments	Screen
		command (d) then drags the right side around randomly to test (figures e-f).		
	Carla: They are still parallel.			
	Researcher: How about dragging the vertex.	Carol drags the top right vertex downward (figures g-i).		 (g) → (h) → (i)
	Carla: But when it's like this (figure i) it's no longer a parallelogram.			
	Researcher: When is it not a parallelogram?	Carla points at Figure i.		
	Carla: Here (figure i).			
	Carol: Try to look obliquely.	Carol shows Carla how to look obliquely, Carla follows		

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Carla: Ah, yes it is (laughs out loud funnily)	her.		

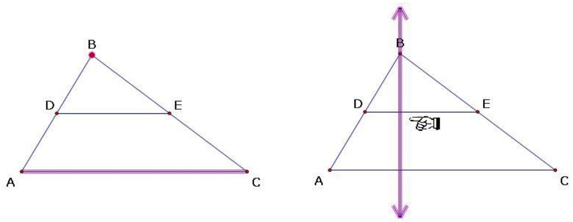
This excerpt shows how the dynamic nature of GSP, especially when geometric shapes can change the appearance under dragging, confuses Carla's understanding of what a parallelogram is. Her adherence to the construction in the paper-and-pencil environment, where most of the shapes are deliberately presented to be aligned on the horizontal or vertical axis, implants a redundant orientation property of shapes in her perception of a parallelogram. This prevents her from accepting the variation of the shape's orientation in the GSP environment (Lines 28-29). Carol (who is usually not very active in the activity) suggests that Carla be free from her rigid frame of reference and look obliquely (Line 33). This suggestion makes Carla realise that the figure actually is a parallelogram when she changes her reference axis.

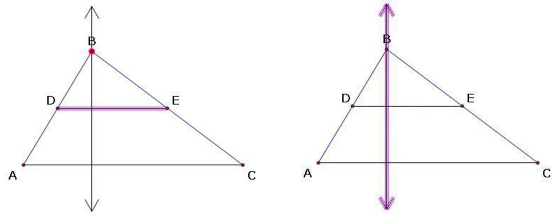
The flexibility of the DGS environment, where the constructed figure can be dragged into various shapes and into different orientations, can also disrupt or mislead the students' reasoning process when the figure's behaviour in the DGS environment does not conform to their prior understanding of the shape. This illustrates the tension in the inter-relationship 2-3-5 where the DGS's unique features influence the students' reasoning process in order to learn Euclidean geometry. Note that the exclusive definition understanding of geometry or the redundant property of shapes understood by the students and adopted by Alex and Carla in the excerpts in this sub-section is not a misconception. Both students have a clear idea of what a parallelogram is, but they just have a different perception of a parallelogram from the way it is portrayed dynamically in the DGS.

10.1.5 Checking with the DGS Tool

Besides verification from a collection of empirical data, some students also adopt a strategy to inductively justify observed geometric property by using DGS tools to check if the statement is true or false. Parallel Line and Perpendicular Line construction commands, as well as the drag-mode, are used to demonstrate and then verify the validity of their observation. This strategy is exemplified and discussed in Alex and Alan's utilisation of the Parallel Line command as a checking tool for Task 3.2 in Excerpt 8.3, and Barbara and Beth's utilisation of the Segment tool to visually check the parallel property in the same task in Excerpt 8.11. This sub-section presents another example of using the GSP construction feature as a checking tool by Brian and Bruce who adopt a different strategy to verify the parallel property in the same task.

Table 10.7 Brian and Bruce's checking strategy

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Brian: If we make B perpendicular to DE and AC, the angles here (at the intersection between DE and the mentioned perpendicular line shown in figure b) will be right angles.		Brian deduces the perpendicular line property assuming that DE is parallel to AC	
		Brian demonstrates by constructing a perpendicular line with AC through point B (figure a).		(a) → (b)
	Bruce: Yeah, and so they are parallel. Researcher: Where are the right angles?	Brian physically points at the intersection in figure b.	Bruce follows Brian's strategy seeing the connection between the perpendicular and parallel properties.	
	Researcher: How do you know that the angles at the intersection are right angles? Bruce: The computer tells us (Brian laughs)		Bruce's response suggests his reliance to	

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Brian: We can also construct another perpendicular line with DE here.		the computer's construction though	
		Brian constructs the second perpendicular line with DE through point B (figure c). The new perpendicular line perfectly overlaps the first one (figure d).	Brian's laughter suggests that Bruce's reasoning is not very sound.	 (c) → (d)
	Brian: Here it is. We got the same line. Bruce: They perfectly overlap.			
	Brian: We won't get two separate lines which mean they are both perpendicular and these lines are parallel.		Brian and Bruce observe the empirical result. Brian deduces from the property of perpendicular lines of a pair of parallel lines.	

Instead of using the Parallel Line command as in Alex and Alan's strategy in Excerpt 8.3, Brian and Bruce opt for the Perpendicular Line command to verify the parallel property of the segments DE and AC. In order to verify the parallel property, they have to deduce the property of the perpendicular line intersecting two parallel lines, where all four angles at each intersection are all right angle. However, Brian defines the property of the perpendicular line (Lines 1-5) by first assuming that DE is parallel to AC, which is the statement that needs to be verified. This strategy can be deemed as an abductive reasoning, where the fact that DE may be parallel to AC is observed. He then deduces the rule of the perpendicular line of a pair of parallel lines in order to demonstrate that such rule works for DE and AC. This abductive reasoning can be categorised as under-coded abduction since the property of the perpendicular line to the two parallel lines is one possible explanation.

After the possible exploratory rule is identified, Brian starts to demonstrate that the rule works for DE and AC by utilising the Perpendicular Line command in the GSP to construct a perpendicular line to AC through point B. When the constructed perpendicular line appears to be perpendicular to the segment DE too, Bruce is convinced that the strategy works (Line 10). The main justification in this strategy is therefore to confirm that the selected rule works for the case in order to verify that the observed parallel property is true. Nevertheless, the researcher challenges the students to verify that the angles at the intersection are truly right angles. Brian then adopts a strategy to construct another perpendicular line to DE through point B, which they observe overlaps the first perpendicular line perfectly (Lines 27-28). The students therefore also use observation by sight to inductively verify that the two perpendicular lines are the same line, resulting in the right angle intersection, thereby confirming the parallel property.

Note that when they are first asked to verify the right angle intersection, Bruce explicitly claims that the perpendicular line is constructed by the computer and hence it should be reliable, especially when it is obvious the perpendicular line is also perpendicular to DE (Line 16). Brian's laugh suggests that he finds Bruce's reasoning funny and not mathematically sound. He then adopts another strategy to construct a new perpendicular line to DE through B in order to confirm the right angle intersection. Brian's reaction shows the subtle attitude he has towards the validity of the construction with the GSP command, since he actually uses it for his process of verification, but he seems to object to the use of the construction as a mere justification of the property.

It is interesting that Brian and Bruce do not use the dynamic feature of the GSP to drag the figure at all throughout this process of verification. They are convinced by the figure in the observed property, in particular the overlapping of the perpendicular lines (Lines 26-27), immediately after construction, without the need to confirm with the drag-test. The reason behind this may stem from the fact that Brian and Bruce's main strategy in the excerpt is to abduct the geometric property of the perpendicular line of two parallel lines to explain the parallel property of DE and AC. The two boys then concentrate on connecting the relevant properties deductively without the use of the dynamic feature. Their application of the GSP is therefore limited to the use of the construction command to help them obtain a checking tool to deduce the parallel property.

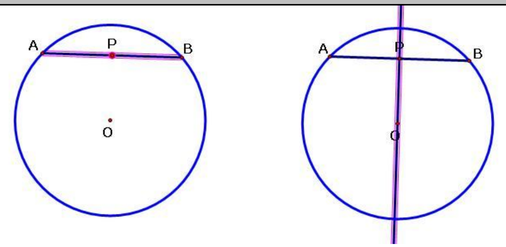
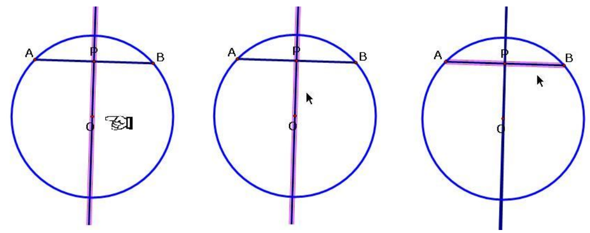
10.2 REASONING WITHOUT THE DGS FEATURE

This section discusses students' reasoning strategies found in the interview which do not directly involve the use of any DGS features. The reasoning strategies are categorised using the discussions in the Literature Review chapter, as well as the identification of the axial codes given in Section 5.7. Though this section is dedicated to the reasoning strategies without the DGS feature utilisation, certain strategies which incorporate the dynamic concept, such as Transformational reasoning, is also included since the students explain this concept without using DGS features to demonstrate.

10.2.1 Reasoning by Sight from a Static Figure

While most students discover the intended geometric property in Task 3 after an exploration phase by dragging the constructed figure, Conrad spots immediately when Colin completes the construction that the perpendicular bisector of the chord passes through the circle's centre which he believes is a mistake. When asked why he thinks the perpendicular line passes point O, Conrad simply responds that the fact can be seen. Nevertheless, he eventually uses the drag-mode to drag the perpendicular line and the chord in order to check. The following excerpt shows Colin and Conrad's response to Task 3.1.

Table 10.8 Colin and Conrad's immediate reasoning by sight

Dialogues	Actions	Reasoning/Comments	Screen
Conrad: Click P, click AB, Construct. Perpendicular Line.	Colin follows Conrad’s instruction (figures a-b).	Conrad instructs Colin to construct the perpendicular Line with the command.	(a) → (b)
Colin: Mm! (satisfied).			
Conrad: But it also passes point O!			
Researcher: Pass point O? Why do you think it passes point O?			
Conrad: Here. It passes point O.	Conrad points at point O (figure c).	Conrad verifies from observation by sight showing that the line actually passes point O.	(c) → (d) → (e)
Conrad: Or not? Let’s try this.	Conrad drags the perpendicular line (figure d) then the chord (figure e) around. The whole figure is just translated.	Conrad then uses the drag-mode to verify the property.	
Conrad: Still passes point O.			

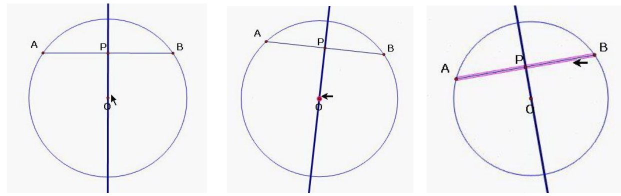
Conrad's remark that the constructed perpendicular line also passes point O at Line 6 does not seem to indicate a discovery, but raises the issue that the instruction requires a line to be constructed through point P, but the line appears to pass point O too. This leads to Conrad's reluctance since he seems to believe that the perpendicular line should only pass point P. When asked to verify why he thinks the perpendicular line also passes point O, Conrad first justifies from observation by sight, physically pointing at point O (figure c) to show that the fact is obvious. He later refines his reasoning by using the drag-mode to verify, but the fact that he clicks on the perpendicular line and the chord, not a point, makes the whole figure translate.

Conrad's reasoning strategy to verify by sight from just a single static instance appears to be even more primitive than the usual inductive reasoning, where the conclusion is drawn from several instances. This excerpt illustrates how Conrad believes in the accuracy of his perception, though he refines his strategy later after being challenged by the researcher. Conrad's attitude towards perception by sight as a warrant for justification is contrary to Alan's rejection of this rudimentary strategy when he performs Task 3.2 with Alex in Excerpt 8.1. Despite its questionable validity, reasoning by sight from a single instance is a common approach adopted by students in this research, especially together with other strategies such as with the construction feature in the DGS exemplified in Excerpt 8.3.

10.2.2 Reference to Coincidence

It is interesting that students sometimes abductively relate the pattern they observe in the geometric exploration tasks to non-mathematical concepts. While performing Task 3.1 of the perpendicular bisector of a chord property, Bruce briefly mentions that the fact that the perpendicular bisector of the circle's chord always passes the circle's centre is just a coincidence. The following excerpt shows this reaction.

Table 10.9 Brian and Bruce's reference to coincidence

Line	Dialogues	Actions	Reasoning/Comments	Screen
		Brian uses the arrow tool to drag point O (figure b) and then the chord AB randomly (figure c).		(a) → (b) → (c)
	Brian: It still passes point O.			
	Bruce: It is already locked.		Bruce adopts a 'locked' mechanism to explain the observed property	
	(5-sec pause)			
	Researcher: Why does it so?			
	Bruce: Because point P is collinear with point O.		Bruce now adopts a mathematic property to explain which turns out to be a circular reasoning.	
	Researcher: How?			
	Bruce: They are on the same straight line.			
	Researcher: So why are they on the same straight line?			
	Bruce: Coincidence? (laughs amusingly) . . Nah, it's not a coincidence. It's like a mathematical rule. How can we prove?		Bruce refers to coincidence as a reason in a teasing way.	

Bruce's series of reasoning strategies in this task shows an interesting justification process, where a range of warrants are used with the observed data to verify the statement. He first refers to a 'locked' mechanism (Line 6) to explain why O never leaves the perpendicular line, regardless of how the figure is dragged. This suggests that Bruce perceives that a control mechanism in the constructed figure organised in a particular way, results in the locked position of point O. Since Bruce does not clearly explain how point O is locked in the figure, this can be interpreted as an under-coded abduction, by proposing one possible explanation of the figure's behaviour. Nevertheless, the incorporation of the 'locked' mechanism may be as a result of Bruce's experience with the parent-and-child relationship in the GSP environment, where a certain part of a figure may move in relation to another.

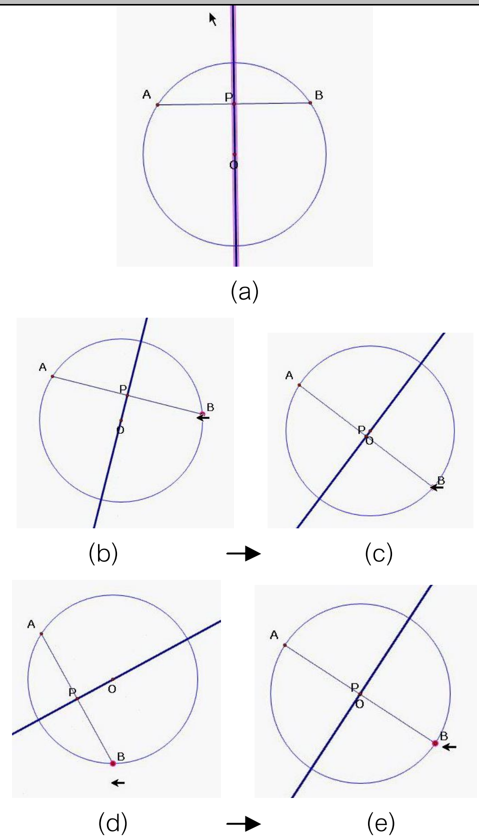
After an abductive reasoning with the 'locked' mechanism, Bruce turns to a more mathematical explanation by referring to the collinear property at Lines 9 and 12. However, this property is actually the statement of the observation that needs confirmation. The reasoning is therefore a circular reasoning.

With persistent challenges from the researcher, Bruce then resorts to a teasing explanation that the observed phenomenon is actually just a coincidence with no governing rule (Line 15). He discards this idea at once before suggesting that it should be a case for a mathematical rule (to be discussed in the next sub-section). Though Bruce's reference to 'coincidence' appears to be merely a joke, it actually highlights the possibility of the recurrence of something unexpected, yet it can happen without any underlying control mechanism. This rejection of causality is therefore a valid explanation, though it also needs further justification of the absence of a governing rule. Such reference to coincidence can be considered an under-coded abduction where the possibility of no causality is proposed as an explanation.

10.2.3 Reference to Mathematical Rules

From the previous sub-section, Bruce explicitly contrasts his reference to 'coincidence' with 'a mathematical rule'. This clearly shows the students' prior experience of other mathematical rules from previous lessons, guiding them to spot the same characteristic in the interview tasks. As well as Bruce, in the same task Alice also considers the observed property to a mathematical rule. The following excerpt shows her response.

Table 10.10 Alice and Alma's reference to mathematical rule

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Alice: We get a perpendicular line and it will pass the centre.		Alice observes that the perpendicular line passes through point O.	
	Researcher: Why does it pass the centre?			
	Alice: Maybe it is a math rule.		Alice abductively reasons that this is the consequence of mathematical rule.	
	Researcher: Do you think it really passes the centre?			
	Both: Yes.			
	(10-sec pause)			
	Researcher: What do you think would happen if we move the diagram?			
		Alice then drags point B downward (figures b-d) then back until point P is on top of point O (figure e)		
	Alice: Here, it really passes! They overlap!		Alice demonstrates that P and O can be overlapped.	
	Alma: Yeah, they overlap!			

In this excerpt, Alice explicitly reasons that the fact that the perpendicular bisector of the chord always passes the circle's centre is a case of mathematical rule (Line 4). However, she does not elaborate what kind of rule she is referring to. Her reference to a mathematical rule as a possible reason of the observed property can therefore be categorised as an under-coded abduction since there is yet no evidence to support this belief. It is possible that Alice connects the special characteristic of mathematical rules learned from previous lessons, especially when a certain property is retained under change, to what she observes from the task. Alice's experience with mathematic rule makes the encounter with similar circumstances in this task unsurprising for her. It may also strengthen Alice's belief that the statement is true, based on her awareness of this concept in the subject of mathematics. Nevertheless, Alice and Alma later utilise the drag-mode to demonstrate that the perpendicular line actually passes point O by placing the midpoint P over point O (Lines 15-16).

Alice's reference to the concept of mathematical rule without a clear identification of its content suggests that the conviction of the claim may not necessarily stem from the rule's actual purpose. She is convinced that the observation belongs to a case of a mathematical rule simply by connecting their characteristic invariants under change conditions. This exemplifies a case where a concept is used as a warrant rather than an identifiable rule. It can guide the students to find a strategy to look for an applicable known in order to verify the statement with an abductive reasoning process.

10.2.4 Transformational Reasoning

The most significant reasoning found in this research is the use of transformational reasoning, where students reason by applying a dynamic rule to explain the observation. This kind of reasoning involves the students observing that when one factor of the figure is changed, some elements are also changed systemically or remain invariant. This mode of reasoning is significantly different from inductive reasoning in various static cases since its gradual change and mechanism are taken into account. Most of the transformational reasoning strategies found in this research remain at an abductive reasoning level. The majority discover the relationship of the changes but are unable to fully explain why they happen in such a way.

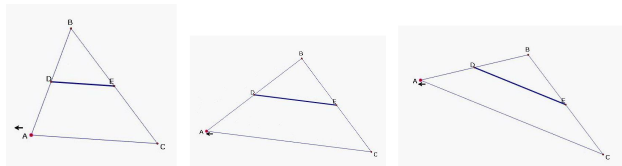
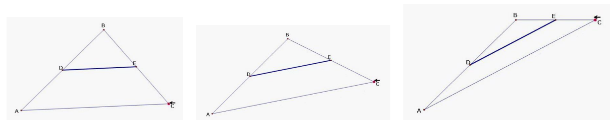
In the exploration of a triangle's midpoint theorem, examples of the students' adoption of transformational reasoning are discussed in Excerpts 8.5 and 8.9, where Alex-Alan and Barbara-Beth use the concept of 'contraction' and 'amplification' to explain the relationship between the segments DE and AC in Task 3.2. These students refer to the process of dilation, where the triangle side AC is either contracted or amplified with the vertex B as a dilation point. Even though their explanations incorporate the dynamic process, the students do not actually use the dynamic feature in the DGS to help them explain such transformation. They simply point at the static figure and verbally describe the change. However, this dynamic characteristic of reasoning can be considered an internalisation of the dynamic environment in the GSP where the students start to think in terms of a continuous process rather than a set of static cases. However, students' transformational reasoning applied in this research still lacks the logical explanation of why the dilation results in the observed parallel property. Their reference to transformational reasoning should therefore be categorised an under-coded abduction, where the dynamic description is proposed as a possible explanation yet to be confirmed.

Unfortunately, the interviewed task designed for this research excludes all available Transform commands in the GSP in order to concentrate on the Euclidean geometry domain of reasoning. Yet the discovery of transformational reasoning as one of the students' strategies suggests that Transformational geometry may also be useful as a part of the explanation to justify a geometric claim. It would be interesting to further investigate how the Transformation features available in many DGS products relate to students' possible adoption of transformational reasoning, especially when they can actually observe the transformation process performed in the DGS environment.

10.2.5 Reasoning from Geometric Definitions

One of the simplest forms of deductive reasoning found in this study is the deduction by reference to the definition of a geometric term used in the statement. Though this reasoning may not provide much further information, it still reflects how students understand the definition of such a term. The following excerpt is taken from Alice and Alma's performance in Task 3.2 after they complete the construction and start to explore the figure.

Table 10.11 Alice and Alma's reasoning from geometric definition

Line	Dialogues	Actions	Reasoning/Comments	Screen
		Students are asked to explore the constructed figure and report their observations.		
	Researcher: How about segment DE? (6-sec pause) Alice: It will get bigger or smaller according to the figure we move so it remains at the middle. Researcher: Anything else?	Alice moves point A randomly (figures a-c).	Researcher needs to guide the student to observe the intended property.	 (a) → (b) → (c)
	Alice: Parallel. Researcher: How? Alice: With AC. Researcher: How do you know they are	Alma moves point C upward (figures d-f).	Alice realises the parallel property.	 (d) → (e) → (f)

Line	Dialogues	Actions	Reasoning/Comments	Screen
20	parallel? Alice: Because it is the same distance. Researcher: What is the same distance? Alice: Here, DE and AC	Alice physically points at the gap between DE and AC.	Alice deduces from the definition of parallel line.	

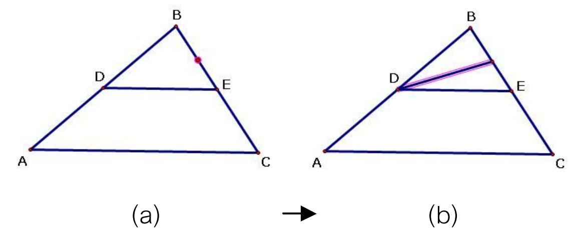
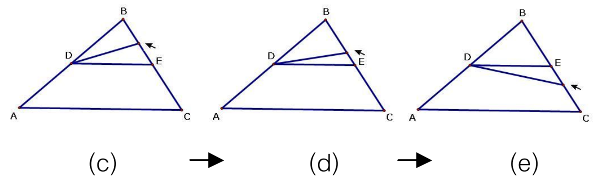
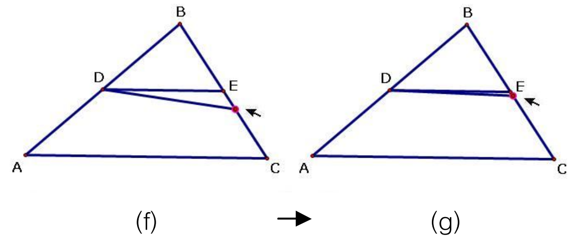
From this excerpt, when asked to justify why they think DE is parallel to AC, Alice simply refers to the definition of parallel lines: two lines that never meet and are always the same distance apart (Lines 19-21). This reasoning is a tautology-like justification process since the reasoning is based on the definition of the element found in the statement, in this case, parallel lines. Though this way of reasoning may not make sense with new information, the absolute correctness of the way things are defined provides an undisputable warrant, based on the common agreement of what parallel lines mean. This form of reasoning may be asserted by the students because of its strength of validity, though they also need to rely on observation by sight to verify the same distance.

Barbara and Beth also adopt this deduction of the parallel line definition in their strategies to verify the triangle's midpoint theorem in this task as discussed in Sub-section 8.2.2. Though these girls never explicitly refer to the property of the parallel line where the distance between the two lines should remain constant throughout, their main strategy is to use the DGS features to inductively demonstrate that the distance between DE and AC is constant. The tautological deduction to the definition of the observed property may not be productive for the reasoning process on its own. Nevertheless, it can be a concrete warrant where visual evidence can be used to verify and justify the property in question.

10.2.6 Reasoning by Modus Tollens

Another example of deductive reasoning which can be considered a contrary case to the tautological deductive reasoning discussed in the previous sub-section, is reasoning by modus tollens, where students use the negative form of the statement to help them justify the observation's validity. Similar to the deduction of the geometric definition discussed in Sub-section 10.2.5, this way of justification does not provide any extra information to the justification process since it simply presents the statement in a negative form. However, it is still helpful for the students to affirm that the statement also works in the contrary case. The following example is taken from Colin and Conrad's performance in Task 3.2, where they are asked to explain why segment DE is always parallel to AC. They use a counter example to demonstrate that if the segment DE does not connect two midpoints, it will not be parallel to the base. They conclude this counter-example immediately after they construct the second segment. Nevertheless, Conrad uses the dynamic feature in the DGS to ascertain this by dragging the new point along the side BC. Here is the excerpt of Colin and Conrad's response to this task.

Table 10.12 Colin and Conrad's reasoning with modus tollens

Line	Dialogues	Action	Reasoning/Comments	Screen
	Conrad: If it is not a midpoint, it won't be parallel, right?	Conrad draws a point on side BC (figure a) and connects it to point D with a new segment. (figure b)	Conrad addresses the negative case of the parallel property.	
	Conrad: See? Because they are midpoint of two lines, if they are not midpoints, they won't be parallel.	Conrad then drags the new point downward (figures c-e)	Conrad uses deductive reasoning in a negative form to explain.	
	Conrad: This one can be moved and when we move this to the midpoint again, it will become parallel.	Conrad moves the new point up (figure f) to point E (figure g)	Conrad uses the dynamic feature to demonstrate the two converse cases.	

From this excerpt, Conrad draws a new segment connecting the midpoint D to a new non-midpoint point on the side BC in order to demonstrate that if the segment does not connect the two midpoints, it will not be parallel to the third side (Lines 7-9). Since this claim is simply a converse of the statement to be justified, this type of reasoning is therefore a deduction by modus tollens. This strategy is also adopted by Alex and Alan as discussed in detail in Excerpt 8.6. However, while Alex and Alan use transformational reasoning with the concept of contraction to explain the converse case, Colin and Conrad actually utilise the dynamic feature of the drag-mode to visually generalise.

From the constructed segment connecting point D and the new point, Conrad drags the new point down, passing point E (Lines 10-11, figures c-d), in order to show that that segment is not just 'a segment' but a representation of a class of segments and that one endpoint is not the midpoint of the triangle. Though the construction of the new segment (which can also be done in the paper-and-pencil environment) is used to demonstrate the statement with modus tollens, Conrad also uses the dynamic feature in the GSP to generalise this segment to 'any' segment that satisfies such property. This reinforces the deductive nature of his reasoning as the 'property' is used as a warrant rather than an observation of a single instance of the new segment. This dragging can be categorised as a bound dummy-locus dragging as identified by Arzarello, Olivero et al. (2002) where students try to drag the semi-draggable point (confined to the side BC) in order to maintain a certain property (non-parallel). It also illustrates the 'descending' process of dragging from theory to concrete evidence where the drag-mode is used to validate the proposed conjecture in modus tollens form. It is interesting that after dragging the new point down to illustrate the generalisability of the modus tollens statement, Conrad finally moves the point back to the midpoint

E (Lines 14-15, figure g) in order to demonstrate the contrast between the statement and the modus tollens. This suggests that Conrad believes D and E to be midpoints of the triangle's sides and is the main factor that makes DE parallel to the triangle's third side. Neither he nor Colin raises the point that DE can also be parallel to AC, even though they are not the midpoints of the triangle's sides as suggested by Alex and Alan.

Note that the utilisation of the dynamic feature by Colin and Conrad in this excerpt demonstrates the modus tollens case of the statement only. Students claim the validity of the modus tollens case immediately after the new segment is constructed, even before they use the mouse to drag the new point. The deduction by modus tollens itself does not involve any DGS feature apart from the diagram, which can also be illustrated in the paper-and-pencil environment.

10.2.7 Standard Deductive Proof

In certain instances, students manage to give deductive reasoning in the verbal proof process in order to justify their geometric construction, and when challenged by the researcher. It seems that the challenge is needed in order to lead students to the thorough proof process since students tend to be satisfied by partial deduction of known rules to provide their explanation. The following excerpt is taken from Brian and Bruce's strategies on Task 4, where they are supposed to construct a square inscribed in a circle, and justify their constructed square. Brian and Bruce construct the square by first drawing a pair of perpendicular diameters with the Straight Line command then use the intersection of the diameters and the circumference to draw the square's four sides. This excerpt shows how Brian and Bruce adopt deductive reasoning to prove that their construction is actually a square when challenged by the researcher.

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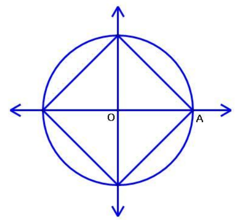
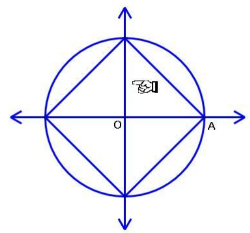
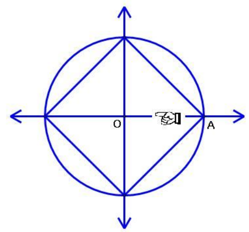
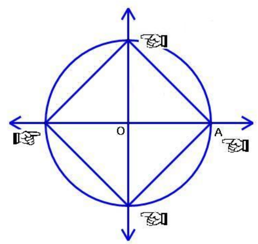
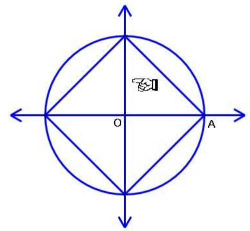
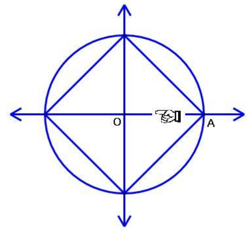
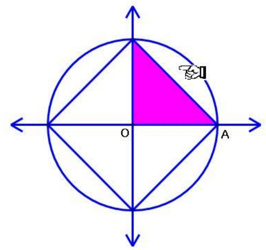
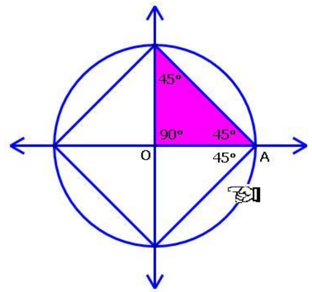
Table 10.13 Brian and Bruce's deductive proof

Line	Dialogues	Actions	Reasoning/Comments	Screen
	Researcher: Why do you think it is a square?			
	Bruce: The intersection point is the centre of the circle and the square. They both share the centre. The centre of the diagram.		Bruce abductively connects the properties of the centre of a square and a circle.	 (a)
	Researcher: Uh huh? So why is it a square?			
	Bruce: Because these two lines (the diameters) divide the circle into four equal parts.	Bruce points at the vertical (figure b) and horizontal (figure c) diameters.	Bruce reasons by symmetric property.	 (b)  (c)
	(3-sec pause)			
	Bruce: Then we connected these four tangent points together so each chord will be equal.	Bruce points at the four intersecting points (figure d).	Bruce's pointing suggests that he means the tangent points as the intersection point.	 (d)
	Researcher: Okay, anything else?			

Line	Dialogues	Actions	Reasoning/Comments	Screen
	(8 sec pause)			
	Researcher: Is it necessary that if all four sides are equal then it must be a square?			
	Bruce: No, it can be a parallelogram or equilateral quadrilateral.		Bruce may refer to a rhombus.	
	Brian: All angles must be 90 degrees.		Brian identifies the property of the square he needs to verify.	 (e)  (f)
	Researcher: Why do you think they are 90 degrees?			
	Brian: Because these two lines are perpendicular so it is 90 degrees and internal angle of a triangle is 180 and this triangle (figure g) is an isosceles.	Brian points at the two diameters (figures e-f). Brian uses his finger to draw the triangle on screen (highlighted in figure g).	Brian deduces from the property of perpendicular line and internal angle of a triangle.	 (g)
	Researcher: Why do you think it is an isosceles?			
	(Both laugh out loud)			
	Brian: I know you must ask me this! It is an		Students are aware of the	

Line	Dialogues	Actions	Reasoning/Comments	Screen
40	isosceles because these two sides are equal. They are radii of the circle, so it is an isosceles. For an isosceles, the base angles must be equal which is 45 and 45 because angle at O is 90. This small triangle is also the same, so 45 and 45 is 90.	Brian points at the two sides of the isosceles (highlighted in figure g). Brian points at the lower left small isosceles (figure h).	importance of reasoning process in the task then deduces from the property of a circle's radii.	 (h) (N.B.: The degree digits are presentational, they do not appear in the diagram)

Before Brian and Bruce adopt deductive reasoning to verify their square construction, Bruce abductively relates the circle's centre to the centre of symmetry of the constructed quadrilateral (Lines 3-4) in order to verify that it is a square. His identification of the centre of the diagram (Lines 5-6) suggests his justification by visualisation by the fact that the two diameters are both the circle's and the quadrilateral's symmetry lines. He later elaborates the symmetric property by saying that the perpendicular lines divide the circles into four equal parts (Lines 10-11). With this appreciation of symmetric property, Bruce and Brian adopt the concept of equality to help them later during the deductive proof process.

After Brian identifies the right angle property of the square he needs to justify at Line 23, he deduces the process of the perpendicular line construction, the property of the sum of the internal angles of a triangle, and four congruent isosceles (due to the symmetry property) to prove that the angle of the quadrilateral is actually 90 degrees. Note that throughout this process of proof by deduction, Brian and Bruce do not use any of the GSP features to help them reason. They merely collect information from the static figure on screen, together with their prior relevant knowledge of the figure to provide input for the deduction process. They even use their fingers instead of the mouse to point at the figure on screen when explaining. This reflects the limitation of the DGS tool to facilitate the deductive proof process. A similar deductive proof process can also be performed with a diagram in the traditional paper-and-pencil environment. This lack of DGS feature utilisation in the deductive proof process conforms to the finding in the research by Connor, Moss et al. (2007), where pre-service mathematics teachers who understand the role of proof actually chose to use DGS less and less when they try to justify mathematical statements with deductive proof.

It is interesting that Brian explicitly acknowledges his awareness of the researcher's role as a reasoning challenger at Line 35 when he expresses that he correctly anticipates the researcher's prompt for a reason. This reaction reflects the student's appreciation of the importance of the reasoning process in the problem-solving task and especially its role as a warrant for the solution.

10.3 DISCUSSION OF LEARNERS' REASONING IN THE DGS

Using the analysis model in Figure 10.1 to analyse the students' reasoning processes in the interview task, the association between each relationship can be outlined. The fact that the designed geometric construction and exploration tasks (including challenges from the researcher) manage to initiate a range of reasoning strategies from the participant students in the main study suggests that appropriate input is needed in order to elicit the students' geometric reasoning process. This affirms the input relationship 2-5 in the analysis model. However, the fact that the DGS is set as a learning tool for students to use in their reasoning process generates a tension between the inter-relationships 1-3-5 and 1-4-5. With DGS set as an available tool, Students are forced to apply their interpretation of the non-dynamic and dynamic features in the GSP to help them reason, though it is still questionable whether GSP provides useful features in the students' preferred reasoning approach. Nevertheless, this particular setting generates a range of varied reasoning strategies from the students. Some of them incorporate the use of DGS features while some of them do not require such features at all.

Two distinct features in the DGS appear to play a significant role in supporting students' geometric reasoning strategies. They are the geometric construction feature and the dynamic feature with the drag-mode. From the excerpts presented in this chapter, students seem to value the precision as well as the property retention of the GSP construction commands as a reliable warrant to help them confirm a geometric property. The fact that the construction commands help students create geometric objects with the desired inherent property that remains forever unchanged provides them with strong validation of their trust in the GSP. This is evident when Barbara and Beth clearly refer to the construction by the Parallel Line command as their verification of the figure's parallel property, and Aaron's assertion that the midpoint P must remain a midpoint of chord AB because it was constructed as a 'midpoint' by the GSP in Sub-section 10.1.2 . The conviction of the reliable property of the construction feature in the GSP is also used as the basis for other reasoning strategies, such as the adoption of the perpendicular property of perpendicular lines constructed by the GSP, verifying that their intersection must be 90 degrees, which provides information for Brian and Bruce's deductive proof process in Sub-section 10.2.7. Moreover, the reliability of the construction feature in the GSP is also creatively used by the students as a checking tool to visually verify the observed geometric property. This is illustrated by three pairs of students' adoption of a segment, a parallel line and a perpendicular line construction to validate the parallel property of the two segments in Task 3.2 in Sub-section 10.1.5. The strategy to use the construction feature in the GSP as a checking tool may be interpreted as the students' way to verify geometric property by measurement, since all the Measure commands were excluded from the designed task by the researcher.

The dynamic feature in the GSP also facilitates the students' exploration and justification process in the geometric situation. As discussed in Sub-section 10.1.1, the dynamic feature helps students observe the variant and invariant properties of components in the figure so as to identify the possible rule. Anna even explicitly claims in Table 10.1 that the property in question can be validated if it is retained when the figure is moved. The dynamic feature also provides students with the flexibility to reason the figure's property, based on a particular case or cases they wish, since it allows them to adjust the figure's appearance to help them reason, such as a collapsed figure, or a right angle triangle as discussed in Sub-section 10.1.3. It also helps students to relate the different cases when the drag-mode can transform them into each other, as shown in Alice and Alma's comparison of a circle's chord and diameter in Table 10.4. Nevertheless, the dynamic feature in the GSP lends itself to an adoption of inclusive definition in geometric shape portrayal in the dynamic geometry environment. This can generate a significant tension when students show an exclusive definitive understanding of geometric shapes, since they can get confused with the unexpected behaviour of shapes in the GSP during their reasoning process, as can be seen in Alex's reaction in Sub-section 10.1.4.

In addition to the direct use of GSP features to support their reasoning, some strategies adopted by the students in this part of the research, also imply the internalisation of experience in the dynamic geometry environment is reflected in their thinking. The justification by transformational reasoning with a dynamic rule (discussed in Sub-section 10.2.4) provides a vivid example. Bruce's description that a certain point is 'locked' in a line in Table 10.2.2 also indicates his awareness of the control mechanism of the parent-and-child relationship in the GSP environment.

This interaction between the GSP environment and students' reasoning strategies is distinctive from the traditional paper-and-pencil environment since it relates to exclusive features of the DGS not found in the paper-and-pencil mode. These are depicted as inter-relationships 1-3-5 and 1-4-5 in the analysis model presented in Figure 10.1.

Apart from the features provided by the GSP, students also draw on a range of other resources in order to justify the geometric property or statement. The prior knowledge and experience of Euclidean geometry, especially in the paper-and-pencil environment plays an essential role in students' performance in geometric construction and exploration tasks. Their prior understanding of geometric definitions and properties provide necessary input for the reasoning process and may contradict with the way GSP portrays geometric shapes, especially when they have a different understanding. Besides their prior knowledge, students also adopt other concepts or information, both mathematical and non-mathematical, as a warrant for their reasoning strategies. These warrants include pure perception by sight, the possibility of coincidence, the presence of an unknown mathematical rule, or the logical proof process. These reasoning strategies, without the use of DGS features, contribute quite significantly to the students' responses to the designed tasks in this research though drag-mode is sometimes used to visually demonstrate the validity of these types of warrant.

Regarding the reasoning strategies, students adopt a number of reasoning types in this part of the research, ranging from inductive reasoning by sight, abductive reasoning to known rules or concepts, to a series of deduction in proof processes. Nevertheless, it is found that a number of abductive and deductive reasoning strategies adopted by the students in this research do not involve the dynamic feature of DGS. There are cases where students discuss their reasoning strategies by simply pointing at the figure on screen without the use of the drag-mode. It seems that the graphic nature of the DGS tool helps to facilitate the type of reasoning that involves observation by perception, such as inductive reasoning. Its genuine role in other reasoning types which require different resources, such as mathematical rules or concepts as in abductive and deductive reasoning, is not yet evident in this research.

The analysis and discussion given in this chapter provides a basis for answering the research sub-question 2 in the next chapter.

11 CONCLUSION

This final chapter reviews all the data analysed and discussed in Chapters 7-10 in order to provide the answers to the research question. The conclusion is drawn from the discussion of each research sub-question in the corresponding chapter, and is broken down into sub-sections, elaborating the key points of each finding. The next section relates these findings to the literature reviewed in Chapters 1-2. The implication of the findings then follows, and the chapter concludes with a discussion of the limitation of this study and suggestions for future research.

11.1 FINDINGS FROM THE RESEARCH QUESTION

The main research question of this study addressed in Chapter 4 of this thesis is:

“What is the relationship between the learner’s higher-order thinking skill of reasoning, and the DGS environment, DGS-based geometric tasks with challenges from a reluctant believer, and how these relate to the learner’s acquisition of Euclidean geometry knowledge?”

Based on the data collected and analysed in the previous chapters, three main findings can be identified. They reflect the tensions between the four main entities: Learner, Euclidean Geometry, Learning Activities and DGS, presented in the model of study in Figure 3.5 in Chapter 3 and further elaborated and modified into an analysis model shown in Figure 6.3 in Chapter 6. These three findings are as follows.

Finding 1: There is a tension between: 1) the designed geometric construction/exploration tasks which aim to orient learners' reasoning toward deductive reasoning; and 2) the inductive reasoning nature of the concrete graphic presentation of Euclidean geometry in the DGS environment provided as a learning tool. There is also a conflict between: 1) the learner's practical view of Euclidean geometry; and 2) the more theoretical approach of Euclidean geometry pedagogy tradition.

Finding 2: Learners' experiences with the traditional paper-and-pencil environment and its relation to Euclidean geometry plays a significant role in their interpretation of the DGS environment, especially when DGS adopts digital flexibility to portray geometrical objects differently with an additional function to support the dynamic capability.

Finding 3: Due to the tensions stated in the first two findings, learners tend to adopt abductive reasoning as their main reasoning strategy to solve the task by posing a hypothesis from their observation, and then collect knowledge, information and evidence from various sources in order to inductively and deductively verify the hypothesis through the process of triangulation. The way the tasks are set challenges them to convince themselves, their colleague and the researcher.

Each of these findings is discussed in detail in the following three sub-sections.

11.1.1 Learners' Different Views of Euclidean Geometry and the Tasks

This finding stems from the research sub-question 3 where the approach and rationale for the tasks are designed to enable learners to use reasoning strategies to acquire Euclidean geometry knowledge. It also reflects the inter-relationship 2-5-6 in the analysis model.

In Chapter 7, the whole process of task design is outlined with the main pedagogical objective being to permit the learners to acquire Euclidean geometry through the process of reasoning. The activities are divided into two parts. The first part: Task 1 allows the learners to explore the DGS commands by themselves. The second part: Tasks 2-4 challenges them with geometric construction and exploration tasks. The rules in the geometric exploration tasks are taken from Euclid's *Elements* and the learners are expected to acquire the knowledge through the process of verification by reasoning.

Though the learners are given freedom to reason in their own way, the researcher's intention is to challenge them to provide stronger reasoning, orienting to deductive reasoning, as traditionally incorporated in Euclid's *Elements*. This brings about a strong tension between the design tasks' learning objective and the way learners actually view Euclidean geometry, especially when they regard Euclidean geometry as a practical subject to help them draw or construct meaningful diagrams. The fact that the DGS portrays Euclidean geometry in the form of graphic representation, similar to the paper-and-pencil environment, complicates the interaction between learners and the governing geometrical rules they are supposed to learn, which are presented through the figure's behaviour. The dynamic feature in DGS may help learners to spot the invariant property when the figure is dragged, but the concrete nature of DGS' Euclidean geometry portrayal lends itself to the inductive way of reasoning rather than deductive.

Analysis in Sub-section 9.1.2 shows Aaron's doubt in the purpose of constructing merely a point, a segment and a straight line in the DGS, as it seems meaningless to him unless they are used to construct some meaningful figures. While Sub-section 9.2.1 shows the way Charles treats DGS as a drawing programme, where he adopts the drag-mode as a facilitating tool to move, adjust the size, and change the orientation of the constructed figure into the desired shape. Sub-section 9.1.5 also gives some examples of how participant students regard the benefit of the DGS tool. Most of them use the DGS as a tool to construct meaningful figures such as geometric diagrams or creative drawings. This reaction suggests that students in this research view DGS as a drawing tool with an extra-feature of dynamic movement, which can help them construct an attractive and meaningful figure. Such notion strongly contrasts to the rationale of the researcher who treats DGS as a learning tool which may help students learn theoretical Euclidean geometry through reasoning.

The gap between the role of DGS as a learning tool, as identified by the researcher, and the more practical interpretation of the DGS used by the learners, leads to awkward interaction between the learners and the researcher during the interview. Learners are challenged towards the deductive reasoning process of justification to conform to the traditional way of learning Euclidean geometry, while they struggle to collect concrete evidence inductively from DGS, partly because it is given in the research as a tool.

Nevertheless, the designed tasks, as well as the reaction of the researcher, still manage to elicit a range of students' reasoning strategies, which finally convince them to believe in the hypothesis, as can be seen in various excerpts in Chapters 7-10. The tasks developed and used in this research are therefore satisfactory to a certain extent in encouraging the learners' reasoning and ability to learn Euclidean geometry.

11.1.2 Unique Portrayal of Euclidean Geometry in DGS

The research sub-question 1 which asks how learners use their geometry knowledge to reason as to how the way DGS portrays Euclidean geometry leads to the second finding of the unique portrayal of Euclidean geometry in the DGS, and its distinction from the paper-and-pencil mode. These inter-relationships are portrayed in 1-3-5, 1-4-5 and 3-4-5 connections in the analysis model. According to the data analyses in Chapter 9, students persistently refer to their knowledge of Euclidean geometry in the paper-and-pencil environment in order to make sense of how DGS works. Sometimes their adherence to the paper-and-pencil environment and its familiarity prevents them from understanding or accepting the different ways DGS portrays Euclidean geometry, especially dynamically.

Examples can be taken from data analyses in Sub-section 9.1.1, where Charles-Chloe and Alice-Alma do not realise the infiniteness of the plane portrayed in DGS, which they can explore with scroll-bars. Also in Sub-section 9.2.2, Bruce and Brian are confused with the infinite-length ray portrayed in DGS which is different from the paper-and-pencil tradition where an arrow head is used to denote infiniteness. This computerised portrayal of geometrical objects where their infinite property can be more accurately displayed may also confuse students in the DGS environment. Such unfamiliarity can affect the way the student reasons to justify a certain task. This can be seen in Sub-section 10.1.2, where Beth does not accept the constructed parallelogram with infinite parallel lines in Task 2, which looks obviously different from a parallelogram in the paper-and-pencil environment, even though her process of construction using the Parallel Line command is correct.

Students' adherence to the exclusive definition of geometric shapes in the paper-and-pencil environment, where a particular kind of shape can be uniquely constructed may cause a conflict with DGS' portrayal of geometric shapes, where the inclusive definition of geometric shape is advocated, i.e. a square is also accepted as one special form of a rectangle. This conflict may mislead the students' justification process in the DGS environment as happens in Alan's case in Sub-section 10.1.4, where he does not accept the correctly constructed parallelogram, since it can be adjusted into a rectangle. On the contrary, acceptance and awareness of the inclusive definition of geometric shapes in the DGS environment can also help students in their reasoning process. This can be seen in Excerpt 8.8 of Sub-section 8.2.2, where Barbara chooses to reason the triangle midpoint theorem from a right angle triangle before trying to generalise it to any triangle.

Apart from the different ways DGS portrays geometric shapes from the paper-and-pencil environment, students also encounter the inevitable programme features to allow the constructed figure to move dynamically. This includes the non-geometrical functional object, such as the mandatory point on a circle's circumference, which allows the user to adjust the size of the triangle or points on a straight line to change the orientation. Though most students can make sense of these additional functional objects, some students still get confused by their presence as in Bruce and Brian's case in Sub-section 9.2.2.

The parent-and-child relationship is another necessary mechanism in DGS in order to allow the dynamic manipulation of the constructed figure for students to generally make sense of and explain the manner of the relationship. They appear to realise that this feature stems from a programme control part and has nothing to do with geometry as presented in Sub-sections 9.1.3 and 9.2.3. Nevertheless, some students may misinterpret the order of dependency of the parent-and-child relationship as can be seen in Aaron's response to Task 3.1 in Sub-section 10.1.2. In this task he misinterprets that the perpendicular line follows the centre O, when in fact it moves in relation to the midpoint P. This misunderstanding can disrupt the student's reasoning process, especially when multiple elements move along simultaneously, leaving little clue of their inherent relationships.

These alien elements in DGS, help it portrays Euclidean geometry more accurately and dynamically, distinguish this environment from the traditional paper-and-pencil mode. The learners' ability to make sense of how DGS works therefore relies upon their prior knowledge of geometry, and their experience with the paper-and-pencil environment, as well as understanding of how DGS constructs and controls the figure. The expectation that DGS should display geometrical construction in exactly the same way as in the paper-and-pencil environment can confuse students when using the software and may affect their utilisation schemes in the geometric exploration and verification tasks.

Since the basic rationale of DGS development is to provide the graphic portrayal of Euclidean geometry in the paper-and-pencil environment in a dynamic mode, all the tensions between learning Euclidean geometry theoretically and deductively in the concrete paper-and-pencil mode remains in the DGS environment. The absence of students' natural tendency to adopt deductive reasoning to justify the observed hypothesis without a push from the researcher reflects this gap. The intention to provide the dynamic platform of the traditional paper-and-pencil mode in the DGS can both support and confuse the students' learning due to these extra features.

11.1.3 Learner's Reasoning Strategies and the Role of DGS

The second research sub-question, which looks at the reasoning strategies learners adopt in geometric construction and exploration tasks in the DGS environment, provides the main focus of this research. This is portrayed in inter-relationships 1-3-5, 1-4-5 and 2-3-5 in the analysis model. The learners' pattern of reasoning strategy in this research is most vividly illustrated in Chapter 8, Task 3.2 which provides two continuous cases of students' reasoning in a particular task. From these excerpts, the main reasoning strategy students adopt during the task is abductive reasoning. They first pose a hypothesis, based on the observed invariant property of the figure under dragging, before collecting knowledge, information and evidence to inductively and deductively verify such hypothesis.

During the process of verification of the hypothesis from abductive reasoning, the students' main strategy is to collect as much varied knowledge, information and evidence as possible to help them in the reasoning process. From the analyses presented in Chapter 8, it can be seen that students do not value a particular kind of reasoning more than others, but they are convinced when two or more different approaches of verification lead them to the common conclusion, and this can be interpreted as a process of triangulation. This is evident in the students' reactions to Task 3.2 in Section 8.2, where Alan is first convinced of the parallel property of a triangle midpoint theorem after he and Alex: 1) observe the retention of the property under dragging; and 2) use the Parallel Line command as a checking tool. Also when Beth is certain about the property when: Barbara 1) uses the drag-mode to adjust the figure and check the distance between DE and AC; 2) applies the concept of constant ratio from Pythagorean theorem lesson; and 3) uses a Segment tool as a checking tool to demonstrate the constant distance between the DE and AC. This process of triangulation shows a different path of mathematical verification from the deductive reasoning traditionally valued in school mathematics (though it is subjected to the posterior analytics problem where students can either accept or not accept the agreed axioms or postulates). It is an approach that has not been studied much in mathematics education research, since most are directed to studying the deductive proof process.

The process of justification by triangulation is also a result of the research setting where students work in pairs under the persistent challenge of the researcher. This welcomes the process of argumentation where students need to discuss their own reasoning strategies with their colleague and the researcher. It reflects the three stages of argumentation proposed by Mason, Burton et al (2010), emphasising that the reasoning process is not just an individual experience, but it also has a societal dimension which usually occurs in the mathematics community.

Though students show no obvious validity preference to a particular type of reasoning, the prevalent reasoning type found in this process of verification is inductive reasoning, where students use the observed evidence of the diagram in DGS to support their claim. This could be the result of the concrete nature of the DGS' portrayal of Euclidean geometry, which lends itself more to inductive reasoning, as discussed in Sub-section 11.1.1 above.

Deductive reasoning can also be found in the justification process of abductive hypothesising, though the deduction mostly refers to prior geometrical knowledge of definitions and rules, rather than adopting the dynamic feature in DGS. Deduction in formal proof can also be found but only after a sustained challenge from the researcher. It is notable that the dynamic feature is not used at all during the deductive proof process.

Students' inductive reasoning adopts the DGS features with direct reference to the retention of the property during dragging as verification. The students adopt construction commands in DGS such as Parallel Line and Segment construction as checking tools to visually verify the hypothesised property, and construct additional cases in the same figure, in order to relate the situation in static and dynamic mode when one construction is in motion. The roles of DGS in inductive reasoning are therefore the provision of exploratory tools, where an invariant property can be observed under the figure's motion, and the provision of an independent and reliable entity which can be used to check the property in question in order to triangulate with a different approach. This reflects how learners treat the DGS as a trustworthy tool they can use to confirm the theory. The fact that it is a finished and supposedly approved software package may convince students of its authority and accuracy even if it sometimes confuses them.

While students show no validity preference to any single type of reasoning, some of them explicitly express the effective reaction to some of the strategies or reasoning, especially when they feel that such strategy or reasoning does not seem suitable as a valid solution or answer to the task. This can be seen from Beth's objection to Barbara's use of a segment as a checking tool in Excerpt 8.11, by claiming that it is too easy, and from Brian's amusing laugh at Bruce's inference to the computer's reliability in Sub-section 10.1.5, implying that the reasoning is too straightforward. This shows another dimension of how students assess each reasoning or strategy apart from its logical conviction.

From the chosen setting of the task-based interview, students appear to adopt a variety of 'warrants' in response to the challenge by the researcher. These warrants range from visual observation in both static and dynamic circumstances, the generalisibility of a certain property of multiple cases, conformation to known geometric rules or definition and the validity of modus tollens; the applicability of the transformational process in transformational reasoning to a non-mathematical warrant such as coincidence or the reliability of computer software. These warrants show the broad source of conviction students may use to verify a hypothesis, and some of them may not directly relate it to a mathematical concept.

Apart from justifying the observed property of the geometric exploration tasks and justifying the construction in the geometric construction tasks, some students in this research also express the need to justify the purpose of each DGS command, as well as the software itself, in order to understand why they should use it as a tool. This is evident in Aaron's incomprehension of the purpose of the Point tool in the GSP, and his utilisation of the Straight Line command to construct a pair of parallel lines by sight in Sub-section 9.1.2.

To summarise, the major role of the dynamic feature in the DGS in students' reasoning strategies is to help them verify the statement in geometric exploration tasks inductively in a different way than in the paper-and-pencil environment. It does not have a direct role in students' deductive reasoning strategy as anticipated by the designed task.

11.2 IMPLICATIONS FOR RESEARCH AND PRACTICE

Though this research adopts interpretivism as its epistemological approach and all the research findings are based on individual interpretation, inviting debate and discussion with other viewpoints through the process of moderation, these findings still reflect some common traits of students' reactions which can possibly occur in other circumstances. From the three findings presented in the previous sections, it can be seen that students interpret Euclidean geometry as portrayed in the dynamic geometry environment significantly differently from the more familiar paper-and-pencil environment. Most students interpret Euclidean geometry in the DGS environment based on comparison with the paper-and-pencil environment, generating a tension when DGS portrays geometric objects differently from those with which students are familiar. This suggests that exposing students to two different versions of Euclidean geometry portrayal may easily lead to confusion. It might be helpful for the students if they are introduced to Euclidean geometry through a single environment. This implies that students may appreciate the concept of Euclidean geometry more faithfully in the DGS environment without prior experience of the paper-and-pencil mode. It is therefore interesting to investigate how students perceive Euclidean geometry through the DGS environment in the absence of paper-and-pencil experience.

This research also points out that the graphic nature of the DGS environment lends itself to students' inductive and abductive approaches of reasoning rather than deductive, regardless of how the task is designed. Mathematics educators should, therefore, be aware of the beneficial role of the DGS and try to exploit its usefulness. The research shows that deductive reasoning requires abstract data of geometric rules or properties to deduce, which is not explicitly present in the DGS environment. DGS does not accommodate deductive reasoning in its own right, and needs some extra elements such as a persistent challenge from others to achieve this type of reasoning. The findings of this research, therefore, oppose the notion found in certain literature that DGS may be used to foster deductive reasoning in the formal proof process.

Despite the clear advocation of inductive and abductive reasoning, the flexibility of the DGS environment facilitates a range of reasoning strategies by the students which can be used as a 'warrant' to believe in certain geometric conjecture. This includes reasoning of multiple cases, continuous dynamic cases, checking by sight and construction and transformational reasoning. This research shows the students' natural reasoning strategies which hardly involve deduction. Mathematics educators should therefore realise these primitive innate reasoning strategies and find a way to challenge them to a more rigorous type of deductive reasoning, provided the axioms are accepted.

The characteristically exploratory benefit of the DGS environment may cause a possible gap between the students' use of the DGS tool to learn geometry and the intended curricular objective where the process of deductive reasoning via mathematical proof is usually valued. The teacher should then play the role of lesson mediator, balancing the use of DGS as a geometrically investigative experimenting tool with the needs of the curriculum. An additional plan may be needed in order to help students relate these exploratory experiences to the curricular learning objectives since DGS may not lead students to achieve such a goal in their own right.

Nevertheless, the students' response to the tasks in this research is limited to the dynamic geometry environment of The Geometer's Sketchpad software only. Using other dynamic geometry software might lead to different outcomes due to different portrayals of Euclidean geometry. Moreover, another researcher may interpret and categorise students' reasoning strategies differently. These findings are therefore tentative. They still require arguments from, and moderation with, different interpreters before they can be applicable to wider circumstances.

11.3 REFLECTION OF THE FINDINGS TO THE LITERATURE

The findings from this research presented in the previous section provide a range of relationships between the DGS environment and the students' learning process, especially in geometric reasoning. These reflect the relevant concepts and other research on a similar topic discussed in the Introduction and Literature Review chapters in a number of different ways. This section aims to relate these research findings to those concepts and studies.

11.3.1 DGS and Higher-Order Thinking

As discussed in Section 2.1, there is no general agreement of what should be defined as higher-order thinking. Nevertheless, reasoning skill is commonly identified by many scholars as one of the higher-order thinking processes. Besides this, reasoning is also categorised by Plato (Lee, 1974) and Piaget (1972) into concrete and abstract reasoning, with abstract reasoning considered higher than concrete reasoning.

On the issue of how DGS can be used to cultivate the learners' higher-order skill of reasoning, as encouraged by The B.E. 2544 Basic Education Mathematics Curriculum (IPST, 2001), the findings of this research indicate that this digital technology can provide a novel platform to incite and broaden the reasoning strategies of learners. Experiencing the DGS environment, students feel the need to justify the raison d'etre of the software as well as its commands. The flexibility of the available features provides a range of digital tools that students can creatively utilise for their reasoning. Besides inviting diverse reasoning strategies, the dynamic feature of DGS also supports the development of the students' reasoning skill from inductive reasoning with concrete evidence to abstract level. The presence of transformational reasoning where students adopt an abstract mechanism of the figure under change to explain a certain geometric property illustrates such development. The abductive reasoning strategies where students need to work between the abstract rule, hypothesis and concrete evidence, also suggests the transition from concrete to abstract reasoning. The adoption of DGS as a learning tool in this research therefore verifies that this digital technology can support student development of the reasoning process.

11.3.2 DGS as a Learning Environment

As discussed in Section 2.5 in Chapter 2, the main educational role of the DGS in this research is set as a learning environment in order to study the authentic impact of digital technology on students' learning. DGS appears to provide a distinctive dynamic Euclidean geometry environment that fosters students' innovative learning, similar to how in the case of 'Turtle Graphics' in Logo programme helps students to learn mathematics in different ways, and reviewed in the same section. The fundamental benefit of the DGS environment is its dynamic feature that allows students to explore the variant and invariant property of the figure under motion. This helps concretise the abstract geometric rule similar to the way in which Logo concretises mathematical concepts such as ratio or negative numbers. This exemplifies how digital technology can be used to help students learn mathematics in a novel way.

Apart from the dynamic feature, DGS also provides other features such as Editing, Display Management and Construction, as well as Measurement and Transformation excluded from this research (though they turn out to be useful for students' chosen strategies). This wide range of features in the DGS provides a learning environment that invites diverse learning trajectories which may not necessarily correspond to the teacher's intention as suggested by Sacristan *et al* (2010) in Section 1.1. In this research the students' wide range of responses and reasoning strategies during a relatively short period of interview time also confirm the benefits of flexibility and efficiency with the integration of digital technology into mathematics education. This conforms to the report of the 2006 International Commission on Mathematical Instruction (ICME) study (Hoyles & Lagrange, 2010) discussed in Chapter 1.

Nevertheless, the fact that the DGS bases its portrayal of Euclidean geometry on the traditional paper-and-pencil environment with enhancing features to support the dynamic possibility, and to portray Euclidean geometry more theoretically accurately, can also confuse the learners. Additional features such as the mandatory object in the constructed figure to help the user control the figure, the parent-and-child relationship or the presentation of infinite objects with scroll-bars, generate a necessary gap for students to become familiar with in order to make sense of how the DGS works. This gap can also alienate some students especially when they adhere to the experience with the traditional paper-and-pencil environment. Learning mathematics in a new digital environment therefore requires a certain degree of knowledge adaptation otherwise it can also simply confuse students with these additional rules. This tension reflects how the process of DG software design may cause confusion to the students when they cannot distinguish the designers' choice of design to mathematics rule as discussed by Balacheff (1993) in Sub-section 2.6.3.

11.3.3 Learners' Interpretations of DGS Features

Though some students in this research originally perceive drag-mode in the DGS as a facilitating tool to help them adjust the constructed figure to draw an attractive picture conveniently, as shown in Charles and Chloe's responses in Sub-section 9.2.1, all of the participant students also use drag-mode extensively during the exploration and justification processes. Nevertheless, the characteristic of the students' use of drag-mode in this research is quite different from the classification by Arzarello, Olivero et al in their research (2002). While the ascending process from drawing to theory and the descending process from theory to drawing in abductive reasoning are also present in this research, the categorisation of drag-mode into 'Wandering dragging', 'Bound dragging', 'Guide dragging', 'Dummy locus dragging' or 'Line dragging' is not very relevant here. The main difference is that the dragging behaviour of students in this research focuses more on changing the appearance of the constructed figure to support their reasoning; such as dragging a triangle into a collapsed figure or into a right angle triangle. It concentrates on the shape as a whole rather than how a single point is dragged as defined by Arzarello, Olivero et al (2002). This may reflect the first level of Van Hiele's original model (1986) where shapes are recognised as a whole by the participant students in this research when they use the drag-mode. Students in Arzarello, Olivero et al (2002) may work at the higher levels where the relationship between the figure's components such as points are also considered. The dragging behaviour of students in this research, brings about an issue regarding the inclusive and exclusive definition of geometric shapes especially when the dynamic feature in the DGS visually accommodates the inclusive definition, i.e. a square and a rectangle are also a special form of a parallelogram. This flexibility while helping students in their reasoning strategy in some instances can also confuse students when they adhere to the exclusive definition, as is evident in Aaron's response in Sub-section 10.1.4. This again highlights the conflicts in Van Hiele's original level 3 of geometric understanding where the relationship among different shapes is established differently by the students and the DGS environment. It also supports the same issue raised by Jones' research (2000) where students encounter similar problems when they are supposed to perform the quadrilateral classification task in the DGS environment.

For the parent-and-child relationship, students in this research understand this feature at such a level that they manage to construct a robust figure that cannot be messed-up, similar to the students in the studies by Healy, Hölzl et al (1994) and Hölzl, Healy et al (1994). However, when they are supposed to interpret this relationship from the construction, students in this research show quite different perceptions. Some of them regard the parent-and-child relationship as 'dependency', as exemplified by Brian and Bruce's interpretation when a circle's centre is deleted in Sub-section 9.1.3, or as 'moving in relation to one another' as interpreted by Carla and Carol in Sub-section 9.2.3, when they use the drag-mode to move part of the figure. Some students, on the other hand, misinterpret the behaviour of the parent-and-child relationship by assuming that the intersection point is the connecting point that makes the whole figure translate along the screen. This is similar to the students in Jones' studies (1996, 1998) in Sub-section 2.8.2, who use the term 'glued together' to explain a similar situation. Moreover, there is also an instance where a student perceives the reverse relationship of parent-and-child in the construction. This can be seen in Aaron's misinterpretation of the figure's behaviour in Task 3.1, when the centre O is moved in Sub-section 10.1.2, where he believes that the perpendicular line follows point O. This reverse interpretation conforms to the findings of Talmon and Yerushalmy's research (2004) as discussed in Sub-section 2.8.2, where the identified dependency is not actually governed by the software.

This research, therefore, confirms that these non-geometrical features incorporated in the DGS are not necessarily intuitive for the students and they need to understand and become familiar with the functions of these features in order to clearly distinguish them from the geometrical contents.

11.3.4 DGS as a Tool to Support Reasoning

As discussed in Section 2.9 in the Literature Review chapter, attempts have been made by many researchers to investigate the possible role the DGS may have to support students' deductive reasoning in a formal proof process, especially a series of report papers in Volume 44 of the Educational Studies in Mathematics journal (2000). In this series of research, it is found that other factors appear to be more essential than the DGS itself to encourage students' deductive reasoning. Marrades and Gutiérrez's research (2000) concludes that students' prior knowledge of geometry plays a very important role when tackling geometric problems requiring a deductive reasoning process in the DGS environment. The findings of the research by Hadas, Hershkowitz et al (2000) suggest that the way the lesson is designed contributes more to the students' success in deductive formal proof than the DGS environment itself. The absence of the dynamic feature utilisation in the formal proof process by Brian and Bruce in Sub-section 10.2.7 in this research also confirms the lack of direct influence of the unique features in the DGS environment on the students' formal proof process. Moreover, most of the deductive reasoning adopted by the students in this research is drawn from their prior geometry knowledge of rules and definitions rather than the DGS features. This suggests that the key benefits of the DGS tool to support the students' reasoning process remains at the inductive and abductive levels of reasoning. The visual presentation of

Euclidean geometry in the dynamic mode helps them justify the geometric property based on concrete evidence observable from the software. This finding conforms to the study by Connor, Moss *et al* (2007), where the pre-service teachers with experience of formal proof, use DGS less and less when they try to justify mathematical statements. It also reflects the gap between the concrete material entity of a geometric drawing and the ideal theoretical object it represents suggested by Laborde (1993) in Sub-section 2.6.3. The significant benefit of the DGS is therefore the provision of a flexible exploratory tool that open doors to a range of inductive and abductive reasoning strategies which may finally lead to a deductive formal proof process under careful guidance from lesson design.

In the same volume of Educational Studies in Mathematics journal discussed in Section 2.9, Laborde (2000) also points out that though DGS provides a flexible platform where students can explore the geometric situation in the way they wish, the more important factor that can lead to a successful proof process is the way the teacher designs the 'milieu' or environment of such task. The execution and outcome of this research obviously complies with this notion, and the process of task design which is treated as a significant entity in this research appears to play an eminent role in the overall process of students' reasoning in the DGS environment. The design of the tasks to include the command familiarisation, the geometric exploration, construction and justification, and the problem-solving sections together with the persistent challenge from the researcher generate a range of reasoning strategies from the students; from inductive reasoning merely by sight, to the deductive formal proof process. The DGS per se may not be a very useful tool for the students' reasoning process without such 'milieu'. The way DGS is used in geometric lessons is also equally important in the students' learning process.

11.4 LIMITATION OF THE STUDY

Due to the time and resource constraints, the researcher conducts this research with just a small group of students and with a set of particular geometric tasks. In such circumstances, the data collected and analysed can reflect only partial interaction between learners, the DGS tool and the designed tasks. The claim of the findings in this research is therefore tentative and can present only the possible scenarios in this chosen setting. Had a different group of students taken part in the research and different sets of tasks been used in the interviews, the results could differ significantly from what has been presented in this thesis. Moreover, the scope of the whole task is also limited by the researcher's choice of commands introduced to the learners. This naturally limits the of range DGS utilisation schemes that students may adopt in their reasoning process. This is evident in the case of Alan and Barbara in Chapter 8, where their reference to transformational reasoning suggests the benefit of Transformation commands also available in DGS, commands which the researcher has already excluded from the tasks.

In terms of the data collection strategy, since the aim of the research is to elicit students cognitive process of reasoning by thought, while tackling the tasks, the impossibility to concretely perceive this, forces the researcher to rely upon the students' conversations and their behaviour during the interview. This limitation has prevented the researcher from fully collecting all the information from the students' thought process. The data collected is therefore inevitably partial and all its analyses are based on the researcher's individual interpretation.

The limited availability of the DGS tools in the Thai language also forces the researcher to use just one DGS tool, i.e. the Thai version of Geometer's Sketchpad in the research. Discussion about each DGS tool available in the market in the Literature Review chapter suggests that different DGS products offer different portrayals and command algorithms from each other. This provides unique representations of Euclidean geometry and students may also react differently, even with the same designed tasks. The findings in this research are therefore based on the Geometer's Sketchpad's portrayal of Euclidean geometry, rather than DGS' portrayal of Euclidean geometry in a general sense. This issue needs to be further discussed amongst mathematicians since there is no standard way for a DGS tool to portray Euclidean geometry.

All these limitations may lessen the generalisability of the claims in this research. However, it still sheds light on the intricate relationships when using DGS as a tool to learn Euclidean geometry through the process of reasoning, especially when different important factors can play significant roles in the students' learning process. This research therefore provides a starting point for investigation into this issue, inviting further research studies with different DGS tools, tasks and settings in order to develop understanding in this area.

11.5 SUGGESTIONS FOR FUTURE RESEARCH

Further research topics relating to this issue may include a systematic comparison of each DGS product available on the market in order to identify common properties, as well as the varying ways DGS can portray Euclidean geometry, and how it manages to accommodate the dynamic feature. Student responses to other domains of geometry also available in DGS packages, such as transformational geometry, coordinate geometry or statistics, are also worth investigation, even in the context of Euclidean geometry pedagogy. Different kinds of tasks, levels of students, DGS packages and different higher-order thinking skills such as problem-solving or creativity may also be studied in a similar setting in order to broaden knowledge of the relationship between technology and learning. Even a repetition of this research with a different interpretation by another researcher may also give an interesting alternative dimension to the results. The area of learning geometry in the DGS environment leaves plenty of scope for the pursuit of further research. It is hoped that this research will inspire other researchers to investigate the relationship between the DGS environment and students' higher-order thinking process of reasoning in a range of interesting aspects in the future.

APPENDIX A:

Ethical Approval Letter and Participant Consent Forms

N.B. The title of study in the following forms is simplified so that students at this age can understand.

ETHICAL APPROVAL LETTER

Mr Alongkot Maiduang,
Department of Education & Professional Studies,
1st September 2010,

Dear Alongkot,

REP(EM)/09/10-64 'Dynamic Geometry Environment and It's Relation to Thai Students' Higher-Order Thinking: Reasoning in Euclidean Geometry.'

I am pleased to inform you that the above application has been reviewed by the E&M Research Ethics Panel that FULL APPROVAL is now granted.

Please ensure that you follow all relevant guidance as laid out in the King's College London *Guidelines on Good Practice in Academic Research* (http://www.kcl.ac.uk/college/policyzone/attachments/good_practice_May_08_FINAL.pdf).

For your information ethical approval is granted until 31st August 2011. If you need approval beyond this point you will need to apply for an extension to approval at least two weeks prior to this explaining why the extension is needed, (please note however that a full re-application will not be necessary unless the protocol has changed). You should also note that if your approval is for one year, you will not be sent a reminder when it is due to lapse.

If you do not start the project within three months of this letter please contact the Research Ethics Office. Should you need to modify the project or request an extension to approval you will need approval for this and should follow the guidance relating to modifying approved applications: http://www.kcl.ac.uk/research/ethics/applicants/modifications.html

Any unforeseen ethical problems arising during the course of the project should be reported to the approving committee/panel. In the event of an untoward event or an adverse reaction a full report must be made to the Chairman of the approving committee/review panel within one week of the incident.

Please would you also note that we may, for the purposes of audit, contact you from time to time to ascertain the status of your research.

If you have any query about any aspect of this ethical approval, please contact your panel/committee administrator in the first instance (http://www.kcl.ac.uk/research/ethics/contacts.html). We wish you every success with this work.

Yours sincerely

____

Daniel Butcher
Research Ethics Administrator

INFORMATION SHEET FOR PARTICIPANTS

YOU WILL BE GIVEN A COPY OF THIS INFORMATION SHEET

Title of Study: Thai Students' Reasoning in the Dynamic Geometry Software Environment

We would like to invite you to participate in this King's College London's PhD in Educational Research project sponsored by The Institute for the Promotion of Teaching Science and Technology (Thailand). You should only participate if you want to; choosing not to take part will not disadvantage you in any way. Before you decide whether or not you want to take part, it is important for you to understand why the research is being undertaken and what your participation will involve. Please take the time to read the following information carefully and discuss it with your guardian. Please let us know if anything is unclear or if you would like more information.

This research on "Thai Students' Reasoning in the Dynamic Geometry Software Environment" aims to study students' learning processes in the computer environment. It will focus on the relationship between student's mathematical reasoning strategies in the geometric software environment. The outcome of this research is expected to clarify how the computer environment can orient the student's reasoning process and to illuminate how technology can be used to support mathematics education.

The research plan is to recruit lower-secondary students aged between 12-15 years old to participate. If you agree to take part, you will be asked to perform a series of geometry tasks with geometry software installed on a computer and will be interviewed simultaneously by the researcher. The tasks and interviews will last between 1 hour and 1 hour 30 minutes and will be conducted outside your classroom time. Subject to your permission, the interview sessions will be video-recorded and your performance on screen will be captured for research purposes. The research will be conducted in school, either in the teacher's room, or a room in the mathematics department. This research should not place you at any greater risk than that encountered in daily school life.

All data to be presented in this research will be strictly anonymous and confidential, i.e. you would not be identifiable as a participant of this research from the report and the data would not be exposed or used for any purpose other than this research. The video clips or stills of you will not be presented in this research or anywhere else. Audio-video recordings, electronic files and screen-captures will be destroyed once data is transcribed and analysed for the final research report.

This research is scheduled to take place between 1st August 2010 and 31st December 2011. It is up to you to decide whether to take part or not. Your decision not to take part will not affect the standard of education you receive. If you decide to take part you are still free to withdraw at any time and without giving a reason. You may also withdraw your data from this research after you have already participated in the interview session. However, once the final report is completed on 31st December 2011 you may no longer withdraw your participation. You can be rest assured that your data will be presented anonymously anyway. If you agree to take part you will also be asked whether you are happy to be contacted about participation in future studies. Your participation in this study will not be affected should you choose not to be re-contacted.

If you do decide to take part you will be given this information sheet to keep and you and your guardian will be asked to sign a consent form. If this study has harmed you in any way you can contact King's College London using the details below for further advice and information:

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เอกสารข้อมูลงานวิจัยสำหรับผู้มีส่วนร่วม

นักเรียนจะได้รับเอกสารข้อมูลงานวิจัยฉบับนี้หนึ่งฉบับ

ชื่องานวิจัย : การใช้เหตุผลในโปรแกรมเรขาคณิตพลวัตของนักเรียนไทย

เราขอเชิญชวนนักเรียนให้เข้าร่วมในงานวิจัยสำหรับวิทยานิพนธ์ระดับปริญญาเอก สาขาการวิจัยทางศึกษาศาสตร์ ของมหาวิทยาลัย King's College London สนับสนุนโดย สถาบันส่งเสริมการสอนวิทยาศาสตร์และเทคโนโลยี (สสวท) นักเรียนควรเข้าร่วมงานวิจัยนี้ ด้วยความสมัครใจ การตัดสินใจไม่เข้าร่วมในงานวิจัยนี้จะไม่ส่งผลเสียแก่นักเรียนแต่ประการใด ก่อนตัดสินใจเข้าร่วมงานวิจัยนี้ นักเรียนควรเข้าใจถึงความสำคัญของงานวิจัยนี้และ นักเรียนจะมีส่วนร่วมในงานวิจัยอย่างไร โปรดอ่านข้อมูลในเอกสารฉบับนี้และปรึกษา ผู้ปกครองของนักเรียน และถามคำถามหากต้องการข้อมูลใด ๆ เพิ่มเติม

งานวิจัยชิ้นนี้เป็นการศึกษาการใช้เหตุผลในโปรแกรมเรขาคณิตพลวัตของนักเรียนไทย มีจุดประสงค์ในการสำรวจกระบวนการเรียนรู้ของนักเรียนขณะใช้โปรแกรม คอมพิวเตอร์ โดยจะมุ่งเน้นถึงความสัมพันธ์ระหว่างกลยุทธ์ในการใช้เหตุผลทางคณิตศาสตร์ กับสิ่งแวดล้อมในโปรแกรมเรขาคณิตพลวัต ผลของงานวิจัยชิ้นนี้คาดว่าจะสามารถแสดง ความเชื่อมโยงระหว่างสิ่งแวดล้อมทางคอมพิวเตอร์ซึ่งอาจส่งผลถึงกระบวนการใช้เหตุผล ของนักเรียนและคาดว่าจะเป็นประโยชน์ต่อการนำเทคโนโลยีมาใช้เพื่อสนับสนุนการเรียน การสอนคณิตศาสตร์

ผู้มีส่วนร่วมในงานวิจัยชิ้นนี้จะเป็นนักเรียนระดับมัธยมศึกษาตอนต้นอายุระหว่าง 12-15 ปี หากนักเรียนตัดสินใจเข้าร่วมในงานวิจัยนี้ นักเรียนจะได้ปฏิบัติกิจกรรมทาง เรขาคณิตในโปรแกรมคอมพิวเตอร์พร้อมตอบคำถามสัมภาษณ์จากผู้วิจัย งานวิจัยจะใช้เวลา ประมาณ 1 ชั่วโมง ถึง 1 ชั่วโมงครึ่ง นอกเวลาเรียนปกติของนักเรียน หากนักเรียนไม่ขัดข้อง การสัมภาษณ์จะถูกบันทึกด้วยกล้องวิดีโอและหน้าจอของโปรแกรมจะถูกบันทึกด้วย โปรแกรมบันทึกหน้าจอเพื่อประโยชน์ของการวิจัย การสัมภาษณ์จะมีขึ้นภายในโรงเรียนใน ห้องพักครู หรือห้องคณิตศาสตร์ การมีส่วนร่วมในงานวิจัยนี้จะไม่ก่อให้เกิดความเสี่ยงใด ๆ มากไปกว่าการใช้ชีวิตในโรงเรียนตามปกติ

การนำเสนอข้อมูลต่าง ๆ ในงานวิจัยจะไม่ปรากฏชื่อผู้เข้าร่วมซึ่งจะถูกเก็บไว้เป็น ความลับ ดังนั้นชื่อของนักเรียนจะไม่สามารถระบุได้จากการเข้าร่วมในงานวิจัยชิ้นนี้ นอกจากนี้ข้อมูลต่าง ๆ จะไม่ถูกนำไปใช้ประโยชน์อย่างอื่นนอกเหนือจากงานวิจัยชิ้นนี้ คลิป วิดีโอหรือภาพนิ่งจะไม่ถูกนำไปแสดงต่อสาธารณชนหรือในรายงานผลการวิจัย ข้อมูลที่

ได้รับการบันทึกทั้งหมด ทั้งวีดีโอ ภาพนิ่ง และไฟล์ต่าง ๆ จะถูกทำลายหลังการวิเคราะห์ข้อมูลเสร็จเรียบร้อยแล้ว

ระยะเวลาของการวิจัยจะอยู่ระหว่างวันที่ 1 สิงหาคม พ.ศ. 2553 ถึง 31 ธันวาคม 2554 การตัดสินใจจะขึ้นกับนักเรียนว่าจะมีส่วนร่วมในงานวิจัยนี้หรือไม่ การตัดสินใจไม่เข้าร่วมในงานวิจัยนี้จะไม่ส่งผลต่อมาตรฐานการศึกษาที่นักเรียนจะได้รับ และหากนักเรียนตัดสินใจเข้าร่วมในงานวิจัยแล้ว นักเรียนสามารถถอนตัวจากการมีส่วนร่วมในงานวิจัยโดยไม่จำเป็นต้องแจ้งเหตุผลได้ นอกจากนี้นักเรียนสามารถถอนข้อมูลส่วนของนักเรียนจากงานวิจัยหลังเข้าร่วมสัมภาษณ์ในงานวิจัยแล้วได้ แต่จะต้องแจ้งให้ผู้วิจัยทราบก่อนวันที่ 31 ธันวาคม พ.ศ. 2554 ซึ่งเป็นกำหนดเสร็จสิ้นการรายงานผลการวิจัยอย่างเป็นทางการ โปรดมั่นใจว่าข้อมูลในงานวิจัยจะไม่มีระบุชื่อจริงของนักเรียน และหากนักเรียนยินดีเข้าร่วมในงานวิจัยนี้ นักเรียนจะได้รับคำถามด้วยว่ายินดีจะเข้าร่วมในงานวิจัยอื่นในอนาคตด้วยหรือไม่ การเข้าร่วมในงานวิจัยนี้จะไม่ได้รับผลกระทบใด ๆ หากนักเรียนเลือกที่จะไม่ได้รับการติดต่ออีก

หากนักเรียนยินดีเข้าร่วมในงานวิจัยนี้ นักเรียนจะได้รับเอกสารข้อมูลงานวิจัยนี้เก็บไว้ และจะได้รับเอกสารยินยอมร่วมในงานวิจัยเพื่อให้นักเรียนและผู้ปกครองลงนามเพื่อส่งกลับยังผู้วิจัย หากงานวิจัยนี้ทำให้นักเรียนเกิดอันตรายไม่ว่าในทางใด นักเรียนหรือผู้ปกครองสามารถติดต่อไปยังมหาวิทยาลัย King’s College London ตามที่อยู่ที่ให้ไว้ด้านล่าง

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STUDENT

King's College London
University of London

CONSENT FORM FOR PARTICIPANTS IN RESEARCH STUDIES (FOR STUDENTS)

Please complete this form after you have read the Information Sheet about the research.

Title of Study: Thai Students' Reasoning in the Dynamic Geometry Software Environment

King's College Research Ethics Committee Ref: REP(EM)/09/10-64

- **Thank you for considering taking part in this research. The person organising the research must explain the project to you before you agree to take part.**

- **If you have any questions arising from the Information Sheet or explanation already given to you, please ask the researcher before you decide whether to join in. You will be given a copy of this Consent Form to keep and refer to at any time.**

- ***I understand that if I decide at any time during the research that I no longer wish to participate in this project, I can notify the researchers involved and withdraw from it immediately without giving any reason. Furthermore, I understand that I will be able to withdraw my data up to the point of publication on 31st December 2011.***

- ***I consent to the processing of my personal information for the purposes explained to me. I understand that such information will be treated in accordance with the terms of the Data Protection Act 1998.***

Participant's Statement:

I ____Class ______________No.______________

agree that the research project named above has been explained to me to my satisfaction and I agree to take part in the study. I have read both the notes written above and the Information Sheet about the project, and understand what the research study involves.

นักเรียน

แบบฟอร์มเข้าร่วมในงานวิจัย (สำหรับนักเรียน)

กรุณากรอกแบบฟอร์มนี้หลังจากได้อ่านเอกสารข้อมูลงานวิจัยสำหรับผู้มีส่วนร่วมแล้ว

ชื่องานวิจัย : การใช้เหตุผลในโปรแกรมเรขาคณิตพลวัตของนักเรียนไทย

หมายเลขอ้างอิงคณะกรรมการจริยธรรมงานวิจัยของ King's College London: REP(EM)/09/10-64

ขอขอบคุณที่พิจารณาเข้าร่วมในงานวิจัยนี้ ผู้วิจัยจะต้องอธิบายรายละเอียดเกี่ยวกับงานวิจัยให้นักเรียนทราบก่อนจะตัดสินใจเข้าร่วม

- หากนักเรียนมีคำถามใด ๆ จากเอกสารข้อมูลงานวิจัย หรือจากคำอธิบาย โปรดถามผู้วิจัยก่อนตัดสินใจเข้าร่วมในงานวิจัย นักเรียนจะได้รับแบบฟอร์มเข้าร่วมในงานวิจัยฉบับนี้หนึ่งชุดสำหรับเก็บไว้
- *ข้าพเจ้าเข้าใจว่าหากข้าพเจ้าประสงค์จะถอนตัวจากงานวิจัยนี้ ณ เวลาใด ข้าพเจ้าสามารถแจ้งให้ผู้วิจัยทราบเพื่อถอนตัวในทันทีโดยไม่ต้องแจ้งเหตุผล นอกจากนี้ข้าพเจ้าเข้าใจว่าข้าพเจ้าสามารถถอนข้อมูลส่วนของข้าพเจ้าจนกระทั่งการจัดพิมพ์รายงานการวิจัย ณ วันที่ 31 ธันวาคม พ.ศ. 2554*
- *ข้าพเจ้ายินยอมให้ใช้ข้อมูลส่วนของข้าพเจ้าสำหรับงานวิจัยตามวัตถุประสงค์ที่อธิบาย ข้าพเจ้าเข้าใจว่าข้อมูลดังกล่าวจะถูกปฏิบัติตามเงื่อนไขของบัญญัติ Data Protection Act 1988 แห่ง สหราชอาณาจักร*

คำแถลงของผู้เข้าร่วมวิจัย

ข้าพเจ้า ____ *นักเรียนชั้น* ____ *เลขที่* ____

ยอมรับว่าผู้วิจัยได้อธิบายรายละเอียดเกี่ยวกับงานวิจัยข้างต้นอย่างน่าพอใจและยินดีเข้าร่วมในงานวิจัยชิ้นนี้ ข้าพเจ้าได้อ่านข้อความข้างต้นและในเอกสารข้อมูลงานวิจัยเรียบร้อยแล้ว และเข้าใจว่างานวิจัยนี้เกี่ยวข้องกับอะไร

GUARDIAN

King's College London
University of London

CONSENT FORM FOR PARTICIPANTS IN RESEARCH STUDIES (FOR GUARDIAN)

Please complete this form after you have read the Information Sheet about the research.

Title of Study: Thai Students' Reasoning in the Dynamic Geometry Software Environment

King's College Research Ethics Committee Ref: REP(EM)/09/10-64

- **Thank you for considering your child taking part in this research. The person organising the research must explain the project to your child before you allow your child to take part.**

- **If you have any questions arising from the Information Sheet or explanation already given to you, please ask the researcher before you decide whether your child should join in. You will be given a copy of this Consent Form to keep and refer to at any time.**

- ***I understand that if I decide at any time during the research that I no longer wish to participate in this project, I can notify the researchers involved and withdraw my child from it immediately without giving any reason. Furthermore, I understand that I will be able to withdraw my child's data up to the point of publication on 31st December 2011.***

- ***I consent to the processing of my child's personal information for the purposes explained to me. I understand that such information will be treated in accordance with the terms of the Data Protection Act 1998.***

Participant's Guardian Statement:

I ____,
a guardian of ______________________________ *Class* _____________*No.*__________

agree that the research project named above has been explained to me to my satisfaction and I agree to have my child take part in the study. I have read both the notes written above and the Information Sheet about the project, and understand what the research study involves.

ผู้ปกครอง

King's College London
University of London

แบบฟอร์มเข้าร่วมในงานวิจัย (สำหรับผู้ปกครอง)

กรุณากรอกแบบฟอร์มนี้หลังจากได้อ่านเอกสารข้อมูลงานวิจัยสำหรับผู้มีส่วนร่วมแล้ว

ชื่องานวิจัย : การใช้เหตุผลในโปรแกรมเรขาคณิตพลวัตของนักเรียนไทย

หมายเลขอ้างอิงคณะกรรมการจริยธรรมงานวิจัยของ King's College London: REP(EM)/09/10-64

- ขอขอบคุณที่พิจารณาให้นักเรียนเข้าร่วมในงานวิจัยนี้ ผู้วิจัยจะต้องอธิบายรายละเอียดเกี่ยวกับงานวิจัยให้นักเรียนทราบก่อนที่ผู้ปกครองจะยินยอมให้นักเรียนเข้าร่วม
- หากผู้ปกครองมีคำถามใด ๆ จากเอกสารข้อมูลงานวิจัย หรือจากคำอธิบาย โปรดถามผู้วิจัยก่อนตัดสินใจให้นักเรียนเข้าร่วมในงานวิจัย ผู้ปกครองจะได้รับแบบฟอร์มเข้าร่วมในงานวิจัยฉบับนี้หนึ่งชุดสำหรับเก็บไว้
- *ข้าพเจ้าเข้าใจว่าหากข้าพเจ้าประสงค์ให้นักเรียนถอนตัวจากงานวิจัยนี้ ณ เวลาใด ข้าพเจ้าสามารถแจ้งให้ผู้วิจัยทราบเพื่อถอนตัวนักเรียนจากการวิจัยได้ในทันทีโดยไม่ต้องแจ้งเหตุผล นอกจากนี้ ข้าพเจ้าเข้าใจว่าข้าพเจ้าสามารถถอนข้อมูลส่วนของนักเรียนจนกระทั่งการจัดพิมพ์รายงานการวิจัย ณ วันที่ 31 ธันวาคม พ.ศ. 2554*
- *ข้าพเจ้ายินยอมให้ใช้ข้อมูลส่วนของนักเรียนในความดูแลข้าพเจ้าสำหรับงานวิจัยตามวัตถุประสงค์ที่อธิบาย ข้าพเจ้าเข้าใจว่าข้อมูลดังกล่าวจะถูกปฏิบัติตามเงื่อนไขของบัญญัติ Data Protection Act 1988 แห่ง สหราชอาณาจักร*

คำแถลงของผู้ปกครองนักเรียน

ข้าพเจ้า ____

ผู้ปกครองของ ______________________________ *นักเรียนชั้น* ____ *เลขที่* ____

ยอมรับว่าผู้วิจัยได้อธิบายรายละเอียดเกี่ยวกับงานวิจัยข้างต้นอย่างน่าพอใจและยินดีเข้าร่วมในงานวิจัยชิ้นนี้ ข้าพเจ้าได้อ่านข้อความข้างต้นและในเอกสารข้อมูลงานวิจัยเรียบร้อยแล้ว และเข้าใจว่างานวิจัยนี้เกี่ยวข้องกับอะไร

APPENDIX B: Interview Tasks

TASK 1: GSP COMMANDS EXPLORATION (20 minutes)

Please try the following commands in the GSP and explain how to use them and what they are for.

TOOLBOX COMMANDS

Command Name	Designed Function
Arrow Tool	Select and/or drag object(s)
Point Tool	Construct a point
Compass Tool	Construct a circle with a centre and a point the circumference
Straight-edge Tool - Segment - Ray - Straight Line	Construct straight objects - Construct a segment - Construct a ray - Construct a straight line
Text Tool	Type texts or label objects

EDIT MENU

Command Name	Designed Function
Undo	Undo the previous operation
Redo	Redo the previous undo operation
Cut	Cut selected object(s) to the clipboard
Copy	Copy selected object(s) to the clipboard
Paste	Paste object(s) from the clipboard
Clear	Clear selected object(s) from the screen

DISPLAY MENU

Command Name	Designed Function
Line Width	Change selected line appearance between ‘dashed’, ‘thin’ and ‘thick’
Colour	Change selected object’s colour
Hide	Hide select object(s)
Show	Show hidden object(s)
Show Labels	Create selected object’s label

CONSTRUCT MENU

Command Name	Designed Function
Point on Object	Construct a point on selected object
Midpoint	Construct a midpoint of a selected segment
Intersection	Construct an intersection point of selected intersecting objects
Segment	Construct a segment connecting two selected points
Ray	Construct a ray from first selected point through the second selected object
Line	Construct a line through two selected points
Parallel Line	Construct a line parallel to a selected straight object through a selected point
Perpendicular Line	Construct a line perpendicular to a selected straight object through a selected point

Please use these commands to draw anything you like.

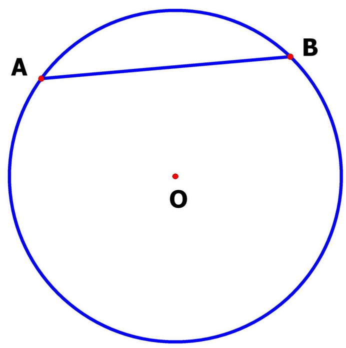
TASK 2: PARALLELOGRAM CONSTRUCTION (15 minutes)

Please construct a parallelogram with the GSP with the condition that the parallelogram must remain a parallelogram when any point is dragged. Please also verify your construction to confirm that it is a parallelogram.

TASK 3: GEOMETRIC EXPLORATION (20 minutes)

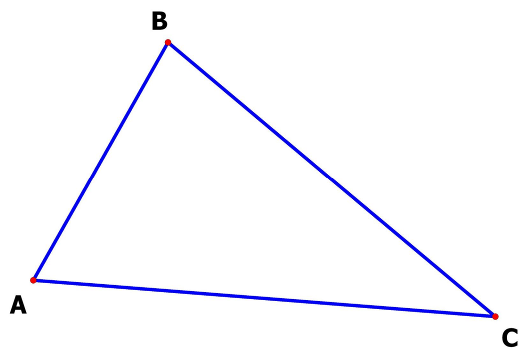
Task 3.1: Exploration of the perpendicular bisector of a chord's property

Construct a midpoint for chord AB, name it P. Construct a line perpendicular to chord AB through point P. Explore the figure by moving any point or line and report your observation.



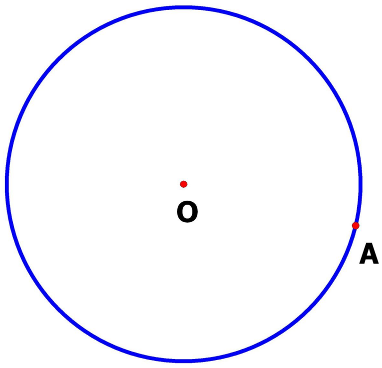
Task 3.2: Exploration of the triangle midpoint theorem

Construct midpoints for sides AB and BC and name them P and Q respectively. Explore the figure by moving any point and report your observation.



TASK 4: PROBLEM-SOLVING (15 minutes)

Construct a square inscribed in the circle O with point A as one of the square's vertices. The constructed square must remain a square when any point is dragged.



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